THE EFFECT OF GROUND SETTLEMENTS ON THE AXIAL RESPONSE OF PILES: SOME CLOSED FORM SOLUTIONS

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***ABSTRACT*:** Simple closed form expressions are presented for the problem of ground settlement effect on the axial response of a vertical pile. The solutions are based on Winkler model, and allow evaluation of soil-pile interaction under a variety of green field ground settlement profile. An example on soil-pile interaction subject to tunnel induced ground settlements is given to demonstrate the use of the proposed solutions.

INTRODUCTION

This note presents several closed form solutions that describe the axial behaviour of a pile subject to vertical ground movements such as by nearby tunnel construction. The Winkler assumption is utilized to derive the closed form solutions and hence they may only be regarded as a rough estimation of the real problem. However, as the solutions can be computed using a hand held calculator or a spreadsheet program, they are considered to be valuable for the initial evaluation before conducting more time consuming numerical analyses. They can also be used to as benchmark for validation of a numerical code based on Winkler model.

The solutions presented in this note are for a pile in homogenous soil. Vertical ground movements may result from numerous sources (e.g. tunnelling induced ground deformation, heave by excavation, consolidation settlement, etc.). Each of these problems is associated with a particular profile of ground settlements and a specific solution may be found. For example, Poulos and Davis (1980) derived a continuum elastic solution for the case of soil settlement that decreases linearly with depth. Loganathan et al. (2001) proposed a solution for piles behaviour due to tunnelling induced ground deformation using boundary element analysis. Although these solutions are very valuable, they are associated with a specific ground settlement pattern and cannot be used for other cases.

The closed form solutions presented in this note utilise general shape functions for the green field ground settlements (i.e. the vertical displacements that would occur in the absence of the pile). These general shape functions may be fitted to the anticipated settlement profile, which may be obtained from field measurements, analytical solutions, or empirical relations. Methods of fitting shape functions are not covered in this note.

FORMULATION

The formulation is based on the following four assumptions: [1] the soil is homogenous, [2] the soil response at any depth is a function of settlement solely at that depth (based on generalized Winkler assumption), [3] no relative slippage occurs between the pile and the soil, and [4] the pile is elastic with constant axial stiffness. The following derivation begins by assuming that there is no base resistance. However, later in this note, the solutions are extended to include this effect.

The differential equation for the Winkler problem of the axial response of a vertical pile is as follows:

$$EA\frac{\partial^2 y(z)}{\partial z^2} - ky(z) = 0 \qquad \text{Eq. 1}$$

where *EA* is the axial rigidity of the pile, *y(z)* is the vertical settlement at depth *z*, *k* is the subgrade modulus, $[k_i]$=F/L^2 (F=force, L=Length). If the pile is loaded locally by a force *P* at a certain depth, ζ, (see Fig. 1), it will experience the following displacements at depth *z*:

$$y(z) = P\cdot f(z,\zeta) = \begin{cases} P\left(\dfrac{\cosh[\lambda\zeta]\cosh[\lambda(L-z)]}{\lambda EA\sinh[\lambda L]}\right) = Pf_1 & \text{when } z \geq \zeta \\ P\left(\dfrac{\cosh[\lambda(L-\zeta)]\cosh[z\lambda]}{\lambda EA\sinh[\lambda L]}\right) = Pf_2 & \text{when } z \leq \zeta \end{cases} \qquad \text{Eq. 2}$$

where $\lambda = \sqrt{k/EA}$.

The governing Winkler equation including the ground settlement effect is:

$$EA\frac{\partial^2 y(z)}{\partial z^2} - k(y(z) - y_{gf}(z)) = 0 \qquad \text{Eq. 3}$$

where $y_{gf}(z)$ is the green field settlement with depth. The effect of green field deformation may be related to a distributed load on the pile, equal to $dp = k\cdot y_{gf}(z)dz$.

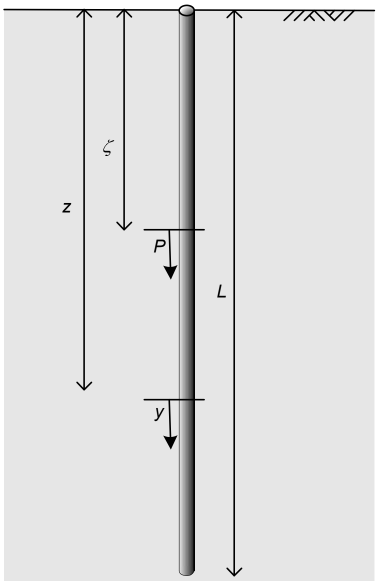

Fig. 1 external loading on pile system

By using the superposition principle of the Winkler assumption, the complete axial behaviour of the pile may be obtained as:

$$y(z) = \int_0^L f(z,\tau)\overbrace{k\cdot y_{gf}(\tau)d\tau}^{dp} = \int_0^z f_1(z,\tau)k\cdot y_{gf}(\tau)d\tau + \int_z^L f_2(z,\tau)k\cdot y_{gf}(\tau)d\tau \qquad \text{Eq. 4}$$

where functions f_1 and f_2 are defined in Eq. 2. This integral can be solved analytically for some green field shape functions given in the following section.

The above formulation assumes that the pile is not restrained at its head. In reality, unless the superstructure is statically determined, this would not be the case. However, if the superstructure is relatively flexible, or that all piles are identical and suffer the same ground movement, this assumption can be considered valid.

CLOSED FORM SOLUTIONS

POLYNOMIAL GREEN FIELD GROUND DISPLACEMENT:

Polynomial fits are very popular; the tools for fitting a polynomial to a given data set are common. A green field ground settlement profile with depth can be approximated by polynomials up to power *q*:

$$y_{gf}(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots + a_q z^q \qquad \text{Eq. 5}$$

An analytical solution of the integral in Eq. 4 using Eq. 5 is possible. Due to the linearity of the problem, the solution may be obtained by superposition of solution for each power in the polynomial series.

$$y(z) = \sum_{n=0}^{q} \phi(a_n);$$

$$\phi(a_n) = a_n \frac{n!}{2} \frac{\cosh[\lambda(L-z)]}{\lambda^n \sinh[\lambda L]} \left[1 - e^{-\lambda z} \sum_{k=0}^{n} \frac{(\lambda z)^k}{k!} - (-1)^n + (-1)^n e^{\lambda z} \sum_{k=0}^{n} \frac{(-\lambda z)^k}{k!} \right] + \qquad \text{Eq. 6}$$

$$a_n \frac{n!}{2} \frac{\cosh[\lambda z]}{\lambda^n \sinh[\lambda L]} \left[\begin{array}{l} e^{\lambda L} e^{-\lambda z} \sum_{k=0}^{n} \frac{(\lambda z)^k}{k!} - (-1)^n e^{-\lambda L} e^{\lambda z} \sum_{k=0}^{n} \frac{(-\lambda z)^k}{k!} \\ + (-1)^n \sum_{k=0}^{n} \frac{(-\lambda L)^k}{k!} - e^{\lambda L} e^{-\lambda L} \sum_{k=0}^{n} \frac{(\lambda L)^k}{k!} \end{array} \right]$$

The above equation may not be suitable for use with a hand held calculator, and it is perhaps more appropriate to use a spread sheet. In many cases, however, high order polynomials are not required; therefore a simple closed form solution using a low order polynomial is presented next.

3RD ORDER POLYNOMIAL

Consider that a green field ground settlement profile with depth can be described using a third order polynomial:

$$y_{gf}(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \qquad \text{Eq. 7}$$

The solution of Eq. 4 using Eq. 7 may be expressed as:

$$\begin{aligned} y(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 \\ &+ \frac{a_1}{\lambda} \frac{\cosh[\lambda(L-z)] - \cosh[\lambda z]}{\sinh[\lambda L]} + 2\frac{a_2}{\lambda^2} \frac{\sinh[\lambda L] - \lambda L \cosh[\lambda z]}{\sinh[\lambda L]} \\ &+ 3\frac{a_3}{\lambda^3} \frac{2\lambda z \sinh[\lambda L] + 2(\cosh[\lambda(L-z)] - \cosh[\lambda z]) - \lambda^2 L^2 \cosh[\lambda z]}{\sinh[\lambda L]} \end{aligned} \qquad \text{Eq. 8}$$

Note that solutions for polynomials of order smaller than three are included in the above expression; that is, these can be obtained directly by omitting terms involving the unnecessary polynomial coefficients.

The pile displacement profile given in Eq. 8 is of the shape:

$$y = y_{gf} + g(\lambda) \qquad \text{Eq. 9}$$

Therefore, function *g(λ)* may be looked upon as a correction function for the case assuming the pile follows the green field settlements. It is also evident from the terms involved in *g(λ),* that the pile simply follows the ground displacement as λ approaches infinite. That is, as *EA* decreases, the pile must follow the green field soil settlement because the pile has no rigidity to alter it.

Since the strains in the pile are of interest for axial force evaluation, the derivative of the pile displacements can be computed as:

$$\frac{\partial y}{\partial z} = a_1 + 2a_2 z + 3a_3 z^2 - a_1 \frac{\sinh[\lambda(L-z)] + \sinh[\lambda z]}{\sinh[\lambda L]} - 2a_2 \frac{L\sinh[\lambda z]}{\sinh[\lambda L]}$$
$$+3\frac{a_3}{\lambda^2}\frac{2\sinh[\lambda L] - 2(\sinh[\lambda(L-z)] + \sinh[\lambda z]) - \lambda^2 L^2 \sinh[\lambda z]}{\sinh[\lambda L]} \qquad \text{Eq.10}$$

***C*OSINE FUNCTION**

In some cases green field ground settlements may fitted with a cosine function. In addition, if a Fourier analysis is conducted on the ground settlement data, the cosine may be used as a fundamental solution for pile response at a given wave length.
Assuming that the green field ground settlement profile with depth is represented by the following:

$$y_{gf}(z) = A\cos(\omega z - b) \qquad \text{Eq. 11}$$

The corresponding solution becomes:

$$y(z) = A\lambda \frac{\lambda\cos[\omega z - b] + \omega\cosh[\lambda z]\dfrac{\cosh[\lambda L]\sin[b] + \sin[\omega L - b]}{\sinh[\lambda L]} - \omega\sin[b]\sinh[\lambda z]}{\lambda^2 + \omega^2} \qquad \text{Eq.12}$$

And the derivative is:

$$\frac{\partial y}{\partial z} = A\lambda^2\omega \frac{-\sin[\omega z - b] + \sinh[\lambda z]\dfrac{\cosh[\lambda L]\sin[b] + \sin[\omega L - b]}{\sinh[\lambda L]} - \sin[b]\cosh[\lambda z]}{\lambda^2 + \omega^2} \qquad \text{Eq. 13}$$

***E*XPONENTIAL FUNCTION**

If the green field ground settlement profile is represented by:

$$y_{gf}(z) = A exp[bz] \qquad \text{Eq. 14}$$

The corresponding solution becomes:

$$y(z) = A\lambda \frac{\lambda e^{bz} + b\dfrac{\cosh[\lambda(L-z)] - e^{bL}\cosh[\lambda z]}{\sinh[\lambda L]}}{\lambda^2 - b^2} \qquad \text{Eq. 15}$$

And the derivative is:

$$\frac{\partial y}{\partial z} = A\lambda^2 b \frac{e^{bz} - \dfrac{\sinh[\lambda(L-z)] + e^{bL}\sinh[\lambda z]}{\sinh[\lambda L]}}{\lambda^2 - b^2} \qquad \text{Eq. 16}$$

INCLUDING THE EFFECT OF TIP RESISTANCE

By recognizing that the resistance force acting on the bottom tip of the pile is equal to $P_b = K_B \lfloor y(L) - y_{gf}(L) \rfloor$ (where K_B is the stiffness of the base in [F/L]), one can use superposition to formulate an equation which includes the base resistance effect. The resultant displacement at the bottom of the pile is equal to:

$$y(L) = \frac{\lambda EA \cdot y^{\text{no base}}(L) + K_B \coth[\lambda L] \cdot y_{gf}(L)}{\lambda EA + K_B \coth[\lambda L]} \qquad \text{Eq. 17}$$

where $y^{\text{no base}}$ is the solution obtained assuming no base resistance (i.e. Eqs. 6, 8, 12 and 15 with $z = L$). Once the displacement at the tip of the pile is determined by Eq. 17, a complete profile of displacement and strain is feasible. This is achieved by superimposing the previously given solution for zero base resistance with Eq. 2 for which $P = K_B \lfloor y_{gf}(L) - y(L) \rfloor$, and ζ=L are substituted.

Although it was stated earlier that the proposed solutions are for homogenous soil, it is also valid for piles which their tip is in a different layer, since K_B may be related to this layer.

EXAMPLE

To illustrate the use of the proposed procedure, let us consider a problem of tunnelling near a pile as shown in Fig. 2. A 5 m diameter tunnel is constructed at a depth of 10 m. A 20 m long pile of 0.5 m diameter exists 6m away from the tunnel centreline. The additional movements that occur in the pile due to the tunnelling are of interest.

We adopt Sagaseta (1987) solution for soil movement due to tunnelling. Fig. 2 shows the green field settlement profile with a tunnel volume loss of 2.5% for the given tunnel geometry). A small heave is observed at depths below 14m. Above this, ground settles with a maximum settlement of about 13mm at a depth of 5m.

The shaft behaviour of the pile is expressed by the Randolph and Wroth (1978) model, of which the load transfer function is $k = 2\pi G / \log(2.5L(1-\nu)/r_0)$, where G is the shear modulus, ν is Poisson's ratio, L and r_0 are the length and radius of the pile. In this example, the shear modulus G is 30MPa and the ground deformation occurs is in undrained condition (ν = 0.5). The EA value of the pile is 5.9 x 10^9 kN (i.e. a concrete pile).

The problem is solved first by assuming zero base resistance. Fig. 3a shows the settlement profile obtained from Sagaseta's solution (solid line) and the best fit using the 3rd order polynomial function (dotted line); the coefficients of the 3rd order polynomial are also given. Fig. 3b presents the resultant pile displacements (solid line) from this fit (dotted line) using Eq. 8. Finally, the additional strain/axial force due to the tunnelling can be computed using the strain profile derived from Eq. 10 and the force profile is shown as a dotted line in Fig. 3c (note that force F in the figure is positive in tension). These forces are additional to the ones originated from the superstructure load. The forces along the pile due to the superstructure may be obtained directly from use of Eq. 2 by positioning the local force at the top of the pile. In Fig. 3c, the "accurate" values (solid line) are based on Sagaseta's ground displacement and not on the 3rd order polynomial fit. They were obtained by high accuracy numerical solution of Eq.3. For this particular problem, the difference between the accurate and 3rd order polynomial fit is 3% at the maximum strain value. With regard to pile displacement, the difference was much smaller, around 0.3%.

Fig. 4 shows the axial force distribution when the base resistance is taken into account. The base stiffness was evaluated as $K_B = 4r_0G/(1-\upsilon)$ (Randolph and Wroth, 1978). When the base resistance is included, similar difference between the accurate and the 3rd order polynomial fit is observed as before. Nonetheless the agreement is very good considering the amount of effort required to establish the values with the current method.

It should be noted that in this specific problem, the tunnelling process induces not only vertical components of soil deformation, but also horizontal. These will result in lateral loading of the pile, and will affect it flexural behaviour, resulting in changes in bending moment values. Flexural behaviour is not covered in the current note.

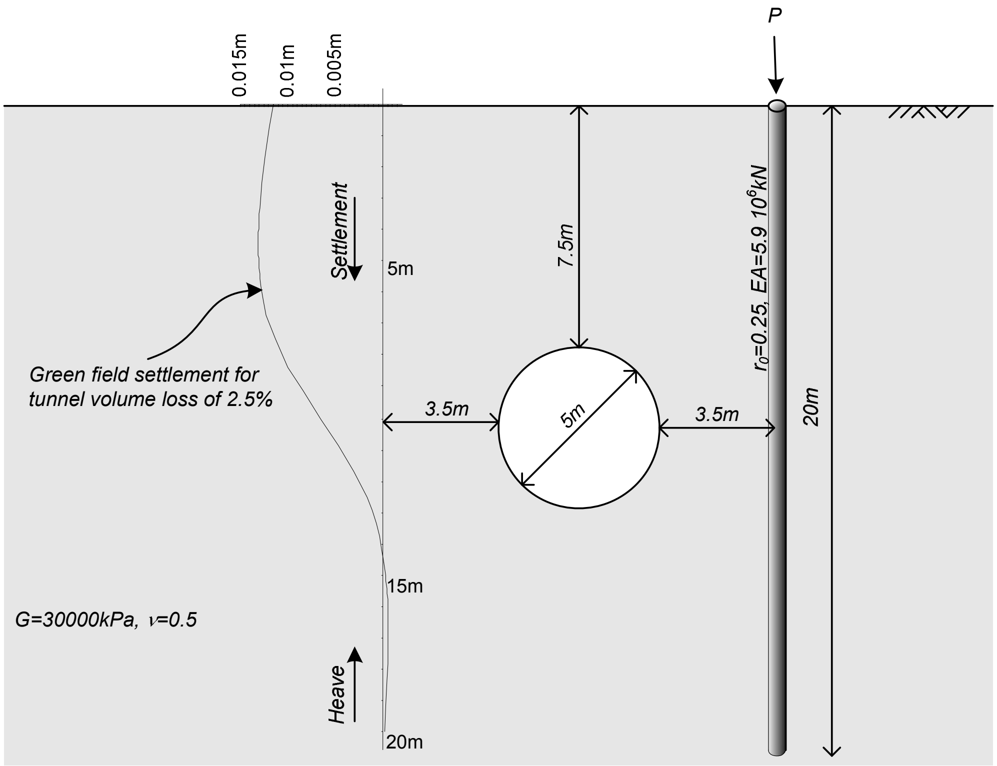

Fig. 2 Schematics of the example problem

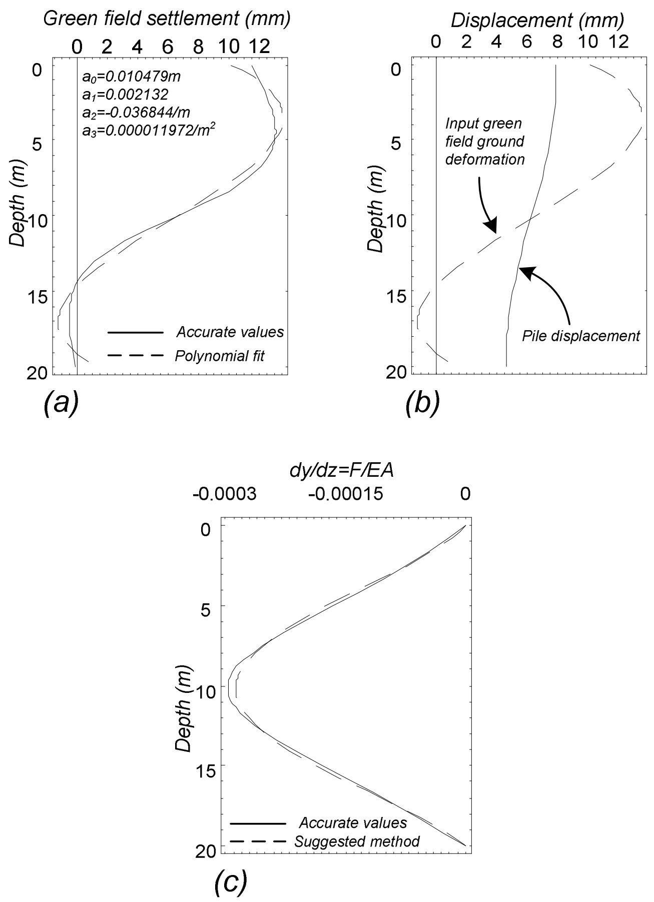

Fig. 3 Results for the example problem

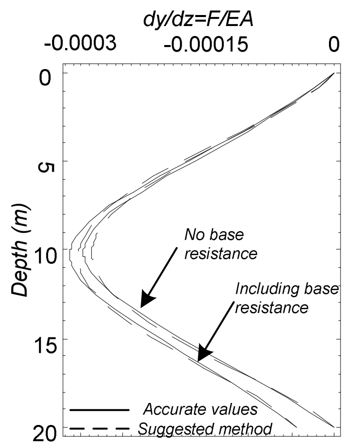

Fig. 4 Results including base resistance

CONCLUSIONS

Closed form mathematical expressions that allow evaluation of the axial response of a pile under the effect of vertical ground movements are given. They are based on Winkler type soil model and therefore can be used in the initial evaluation of the problem before performing more time consuming numerical analysis of soil-pile interaction. The solutions are derived using three different curve fitting functions for the green field ground settlement profile. Each function results in a simple closed form solution that can easily be computed by a hand held calculator or be implemented into a spread sheet program. Using the superposition principle, the

functions may also be used together. An example problem of tunnelling effects on pile was used to demonstrate the use of the present solutions.
The solutions presented in this note refer only to the effect on the axial response due to vertical ground movements. In problems where the green field ground deformation has substantial horizontal component, a bending behaviour of the pile will supplement to the axial one. The current solutions do not cover this issue and further investigation is needed to include this effect.

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