SOIL - PIPE – TUNNEL INTERACTION: COMPARISON BETWEEN WINKLER AND ELASTIC CONTINUUM SOLUTIONS

CUED/D-SOILS/TR 332 (May 2004)

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ABSTRACT

An elastic continuum solution and a Winkler solution of the problem of tunneling effects on existing pipelines are given. A comparison is made between the rigorous elastic continuum solution and a closed form Winkler solution with Vesic subgrade modulus. Although applying the Vesic expression results in the same moments and displacements under external loading in a Winkler system and the elastic continuum, it is found that its use is not necessarily adequate for the problem of tunneling effects on pipelines and may not be conservative due to possible underestimation of bending moment. An alternative expression for the subgrade modulus is provided, resulting in similar maximum bending moments in the Winkler and elastic continuum systems.

Key words: Pipelines, Soil-pipe interaction, Winkler model, Elastic Continuum Solution, Tunnels, Aging infrastructures

INTRODUCTION

One of the challenges facing engineers in the 21st century is the operation and maintenance of aging infrastructure such as pipelines. The current note addresses the effect of tunneling on existing buried pipelines. Fig. 1 shows a schematic diagram of the problem, in which a new tunnel is excavated under an existing pipe. The tunnel excavation generates soil settlement around the pipe causing it to deform. The pipe suffers additional bending moment. The magnitude of pipe deformation and the changes in bending moment depend on the distribution of soil settlement by tunneling at the pipeline level and the relative stiffness between the pipe and the surrounding soil.

The conventional approach for obtaining a solution for this problem utilizes Winkler based models such as proposed by Attewell et al. (1986). In such case, an appropriate subgrade modulus (spring coefficient) needs to be assumed both for linear elastic and nonlinear analyses. In linear elastic analysis the subgrade modulus is usually determined by means of the Vesic (1961) expression (suggested for the current problem by Attewell et al., 1986). Vesic's expression essentially allows a beam on a Winkler foundation to exhibit similar displacements and moments to that of a beam on an elastic half space when loaded with the same load.

The aim of this note is to discuss the validity of the implementation of Vesic's expression to the current problem by comparing to the rigorous elastic continuum solution. This aim is achieved by comparing a Winkler system and an elastic continuum system under the same key assumptions as follows:

[1] A continuous elastic homogenous pipeline line is buried in homogenous soil, **[2]** The pipe is always in contact with the soil, **[3]** The pipe does not affect the tunnel, **[4]** The soil response to loading, at pipe level, is not aware of the tunnel (in the elastic continuum system this relaxing assumption allows us the use of Mindlin's (1936) Green function for vertical load in a semi-infinite half space)**,** **[5]** The pipeline is continuous, and **[6]** The green field soil displacement at the 'pipe' level is described by a Gaussian curve (Peck, 1969) given as

$$S_v(x) = S_{\max} \exp\left[-0.5\left(\frac{x}{i}\right)^2\right] \qquad \text{Eq. 1}$$

where S_{max} is the maximum settlement, x is the horizontal distance from the tunnel centerline, and i is the distance to the inflection point of the green field trough settlement profile.

As mentioned, if a Vesic-Winkler system is loaded with the same load as the elastic continuum system it will exhibit similar bending moment to that of the elastic continuum system. In the current problem, the tunneling effect may be represented as loads on the system related to the soil green field displacements at the pipe level, $S_v(x)$. These loads will generally be different in the Vesic-Winkler and the elastic continuum systems; hence, the use of Vesic’s expression for this problem will generally result in different bending moments in Winkler system compared to those in the elastic continuum system. The magnitude of the difference is presented in this note.

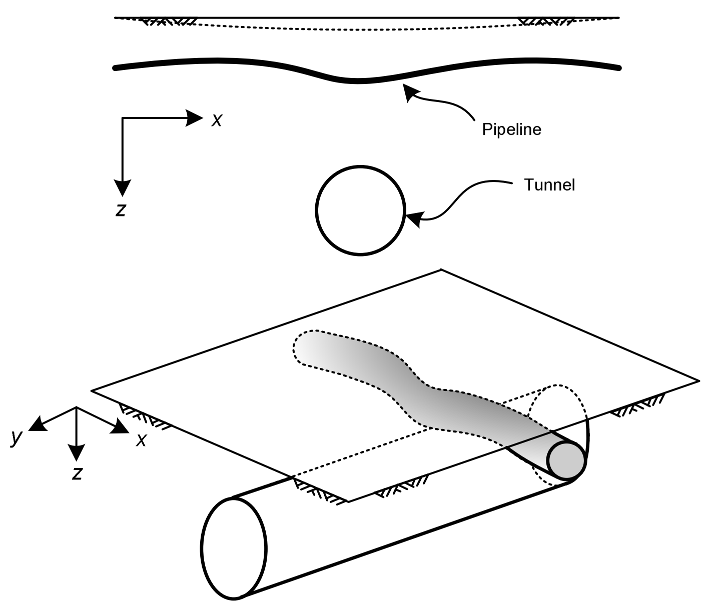


Fig. 1 Schematic representation of the problem

ELASTIC CONTINUUM SOLUTION

The pipe behavior may be represented by the following equation:

$$[S]\{u\} = \{F\} \quad \text{Eq. 2}$$

where $[S]$ is the stiffness matrix of the pipe composed of standard beam elements, $\{u\}$ is the pipe displacements, and $\{F\}$ is a force vector representing the soil loading acting on the beam elements. The pipe is buried within the soil and additional external loads to that of the soil do not generally exist, although they can be added if necessary, e.g. building structure loading.

The soil continuum displacement, u^C, can be represented using a Green function:

$$\{u^C\}_i = \sum_{j=1}^{n} \{f\}_j G_{i,j} \qquad \text{Eq. 3}$$

where $\{f\}$ is the pipe force acting on the soil medium, $G_{i,j}$ is the green function which defines the elastic soil continuum displacement at point i due to unit loading at point j
The summation in Eq.3 can also be written as follows:

$$\{u^C\}_i = \overbrace{\{f\}_i G_{i,i}}^{\{u^{CL}\}_i} + \overbrace{\sum_{\substack{j=1 \\ j\neq i}}^{n} \{f\}_j G_{i,j}}^{\{u^{CA}\}_i} \qquad \text{Eq. 4}$$

where $\{u^{CL}\}$ is the defined herein as local displacement which is the displacement at a point due to its loading solely, and $\{u^{CA}\}$ is additional displacement at that point due to forces acting at different points. Due to assumption **[3]** only degrees of freedom of the pipe need to be considered and i index can therefore be related only to the pipe. Nevertheless, $\{u^{CA}\}$ still involves quantities that result from the tunnel (i.e. j index is still related to the tunnel degree of freedom). However, since it was assumed that the pipe does not affect the tunnel these quantities (i.e. forces) can be decoupled as follows:

$$\{u^C\}_i = \overbrace{\{f\}_i G_{i,i}}^{\{u^{CL}\}_i} + \overbrace{\sum_{\substack{j=\text{first pipe node} \\ j\neq i}}^{\text{last pipe node}} \{f\}_j G_{i,j}}^{\{u^{CAP}\}_i} + \{u^{CAT}\}_i \qquad \text{Eq. 5}$$

where $\{u^{CAP}\}$ is the additional displacement due to forces resulting from soil pipe interaction and $\{u^{CAT}\}$ the additional displacement due to the existence of the tunnel. This decomposition is valid as long as assumption **[3]** holds.
Remembering that the force acting on the soil is the reaction for the pipe, one can define the soil reaction on the pipe from the above equation:

$$\{F\}_i = -\{f\}_i = -\frac{\{u^{CL}\}_i}{G_{i,i}} \qquad \text{Eq. 6}$$

The compatibility relation of $\{u\} = \{u^C\} = \{u^{CL}\} + \{u^{CAP}\} + \{u^{CAT}\}$ is required, and by introducing this and Eq. 6 into Eq. 2 the following relation is obtained:

$$[S]\{u\} + [K^*]\{u\} = [K^*]\{u^{CAP}\} + [K^*]\{u^{CAT}\}$$
$$[K^*]_{i,j} = \frac{1}{G_{i,i}}\delta_{ij} \qquad \text{Eq. 7}$$

where $[K^*]$ is a local stiffness matrix, and is diagonal.

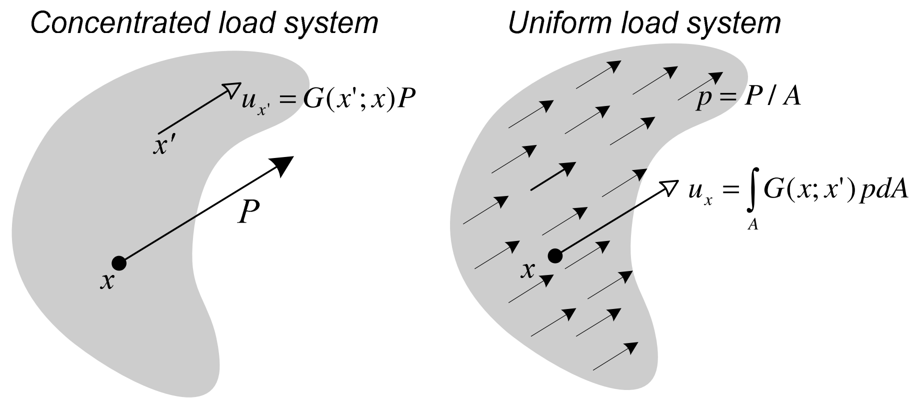
Eq. 5 shows that $\{u^{CAP}\} = [\lambda^* s]\{f\}$ (where $[\lambda^* s]_{i,j} = (1-\delta_{ij})G_{i,j}$; δ_{ij} is Kronecker delta), and with $\{f\} = -\{F\} = -[S]\{u\}$ Eq. 7 becomes

$$[[S]+[K^*]+[K^*][\lambda^* s][S]]\{u\} = [K^*]\{u^{CAT}\} \qquad \text{Eq. 8}$$

which is solved numerically to obtain the elastic continuum solution. $\{u^{CAT}\}$ is the green field displacement.

It is to note that omitting $[K^*][\lambda^* s][S]$ in the above equation results in a Winkler-like model, where the soil reaction acting on the pipe is not affected by the soil response at different locations along the pipe. The term $[K^*][\lambda^* s][S]$ can thus be regarded as an additional term that takes account of continuum effects. This, however, does not mean that the solution obtained by omitting this term is the Winkler solution since the components of $[K^*]$ are different from those which will be constructed using the Vesic (1961) subgrade modulus.

In this study, the Mindlin (1936) solution (Green function) for a point load is used to construct the components of Eq. 8. However, since Mindlin’s solution does not satisfy displacement at the point of loading, a reference displacement value for that point was considered to be the average displacement around the circumference of the pipe. This is identical to assuming a barrel load around the pipe. Actually any displacement at a point due to uniform load is equal to the average displacement, over the same area (or volume) as that of the uniform load, due to an equivalent concentrated load at that point as illustrated in Fig 2. This is due to the reciprocity property of the Green function which states that the response at *x’* due to a delta function at *x* is equal to the response at *x* due to a delta function at *x’*.



By definition G(x';x)=G(x;x'), hence

$$u_{av} = \frac{\int_A u_{x'} dA}{A} = \frac{\int_A G(x';x)PdA}{A} = \frac{P}{A}\int_A G(x';x)dA = \int_A G(x;x')\,pdA = u_x$$

Fig. 2: Explanation of the barrel load identity

To enable a general solution corresponding to different soil and pipe characteristics the results were normalized. The proposed normalization covers all independent parameters and was used to describe bending moment. Fig. 3 shows the computed normalized maximum sagging bending moments occurring at the tunnel centerline in relation to a rigidity factor *R,* defined as $R=EI/Esr_0i^3$ (*EI* is the bending stiffness of the pipe, *Es* is Young's modulus of the soil, and r_0 is the radius of the pipe). The normalized bending moment is defined as Mi^2/EIS_{max}. It should be noted that all elastic analysis values presented in this note assumes Poisson's ratio ν of 0.25, which is an acceptable value for soil under drained conditions. Nevertheless, it was found that the response is not sensitive to the value of Poisson's ratio using the current normalization with *Es* (e.g. difference in bending moments of less than 1.5% between ν=0.25 to ν=0.5, for the complete range plotted).

Fig. 3 also shows the influence of pipe embedment depth, Z, from which it is clear that the embedment depth *per se* has little significance (for later comparison with the Winkler solution $Z/r_0=7$ is chosen). Furthermore, the results, in the range plotted, were found to be practically independent of the ratio i/r_0 when *R* was chosen as a non-dimensional controlling parameter. Fig. 4 provides bending moments along a pipe for different *R* values and shows that the normalized bending moment decreases with increasing R. This highlights the significance of R as can be seen from the difference to the case where the pipe is forced to follow the green field settlement profile (i.e. forcing behavior similar to R=0).

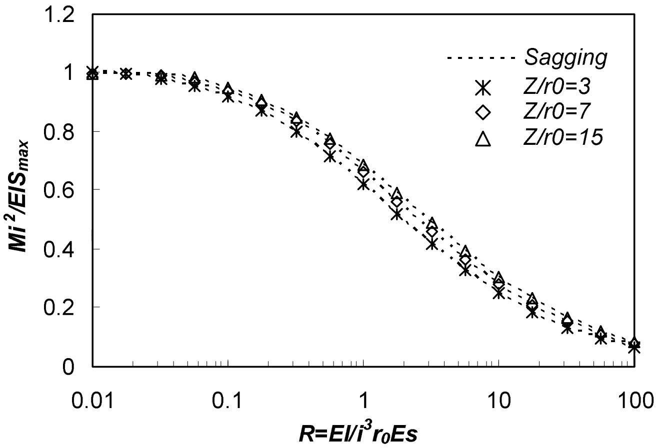


Fig. 3 Maximum sagging moments

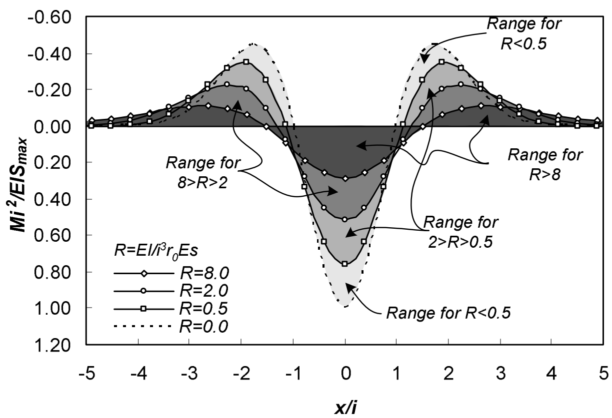


Fig. 4 Normalized bending moment along the pipe, Z/r_0=7

In the following section a closed form solution for the Winkler problem under the same assumptions described earlier is obtained. This solution is later compared to the above elastic continuum solution.

CLOSED FORM SOLUTION OF THE WINKER PROBLEM

The work by Attewelll et al. (1986) is often used to analyze tunnel-pipe interaction. They obtained a numerical solution for the Winkler problem under the aforementioned assumptions and used the following differential equation to represent the pipeline behavior:

$$\frac{\partial^4 S_p}{\partial x^4} + 4\lambda^4 S_p = 4\lambda^4 S_v(x) \qquad \text{Eq. 9}$$

where $\lambda = \sqrt[4]{\frac{K}{4EI}}$, *EI* is the bending stiffness of the pipe and *K* is the subgrade modulus, S_p is the vertical pipe displacement and S_v is the green field soil settlement (i.e the soil settlement at the pipe level if it would not exist). An alternative mechanical system to Eq. 9 is shown in Fig. 5.

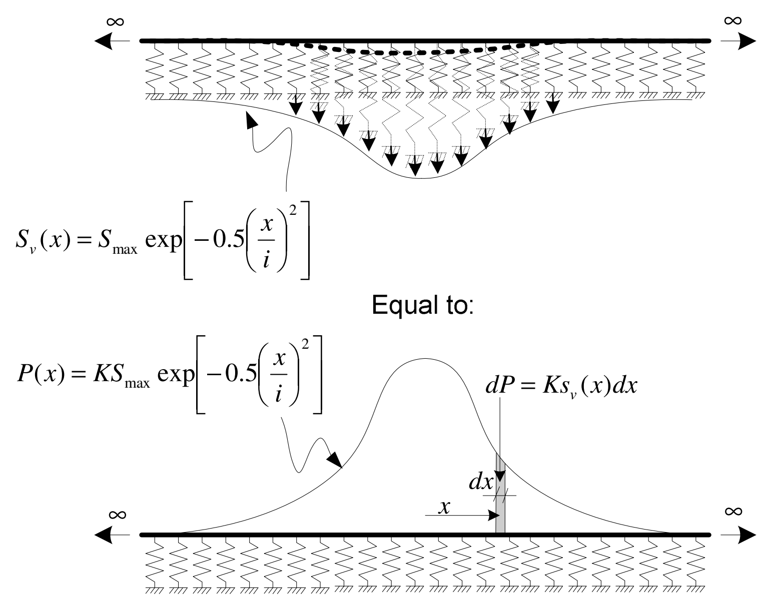


Fig. 5 Mechanical representation of Eq. 9

For an infinite Winkler beam, a concentrated load *P* creates a bending moment *M* of the following magnitude at a distance *t* from the location of the load (after, Hetenyi 1946):

$$M = \frac{P}{4\lambda} \exp(-\lambda t)\left[\cos(\lambda t) - \sin(\lambda t)\right] \qquad \text{Eq. 10}$$

The continuous loading due to the soil trough settlement can be replaced by an infinite number of infinitesimal concentrated loads *dP(x)*, the magnitudes of which depend on the distance from the tunnel centre, *x*:

$$dP(x) = KS_v(x)dx \qquad \text{Eq. 11}$$

The maximum bending moment in the pipe occurs above the tunnel centerline and is referred to as the maximum sagging moment. Using Eq (10), each of the mentioned concentrated loads contributes the following amount to the bending moment at *x=0*:

$$dM(x) = \frac{dP(x)}{4\lambda}\exp(-\lambda|x|)\left[\cos(\lambda|x|) - \sin(\lambda|x|)\right] \qquad \text{Eq. 12}$$

The influence of all infinitesimal concentrated loads is therefore:

$$\begin{aligned} M_{\max} &= \int_{-\infty}^{\infty}\frac{KS_v(x)}{4\lambda}\exp(-\lambda|x|)\left[\cos(\lambda|x|) - \sin(\lambda|x|)\right]dx = \\ &2EI\lambda^3 S_{\max}\int_0^{\infty}\exp\left[-\lambda x - 0.5\left(\frac{x}{i}\right)^2\right]\left[\cos(\lambda x) - \sin(\lambda x)\right]dx \end{aligned} \qquad \text{Eq. 13}$$

Rewriting Eq. 13, a normalized maximum sagging moment can be defined:

$$\frac{M_{\max} i^2}{EIS_{\max}} = 2\lambda^3 i^2 \int_0^{\infty}\exp\left[-\lambda x - 0.5\left(\frac{x}{i}\right)^2\right]\left[\cos(\lambda x) - \sin(\lambda x)\right]dx \qquad \text{Eq. 14}$$

A closed form solution for the above equation is feasible and is equal to:

$$\begin{aligned} &\frac{M_{\max} i^2}{EIS_{\max}} = \sum_{j=0}^{\infty}(-1)^j \xi^{4j}\left(\frac{\sqrt{2\pi}}{(2j)!}\xi^3 - \frac{4^{2j+1}(2j)!}{(1+4j)!}\xi^4 + \frac{\sqrt{2\pi}}{(1+2j)!}\xi^5\right) \\ &\xi = \lambda i \end{aligned} \qquad \text{Eq. 15}$$

Alternatively, the above equation is equivalent to:

$$\begin{aligned} &\frac{M_{\max} i^2}{EIS_{\max}} = \sqrt{2\pi}\xi^3\left\{\cos(\xi^2)\left[1 - 2C(2\xi/\sqrt{2\pi})\right] + \sin(\xi^2)\left[1 - 2S(2\xi/\sqrt{2\pi})\right]\right\} \\ &C(u) \equiv \int_0^u \cos\left(\frac{1}{2}\pi t^2\right)dt;\ S(u) \equiv \int_0^u \sin\left(\frac{1}{2}\pi t^2\right)dt \end{aligned} \qquad \text{Eq. 16}$$

where *C(u)* and *S(u)* are Fresnel Integrals. The limit of the above equation when λi tends to zero is $\xi^3\sqrt{2\pi}$, which is exactly the solution obtained under the assumption of a single concentrated load of magnitude *P=KVL,* where *VL* is the volume loss at the pipe level, equal to $\sqrt{2\pi}S_{\max}i$. That means that as the pipe rigidity increases it feels the soil loading increasingly as a localized loading.

Fig. 6 shows a comparison of the above solution with the numerical values derived by Attewell et al. (1986) to fit their suggested solution. General agreement exists between the numerical values and the current closed form solution.

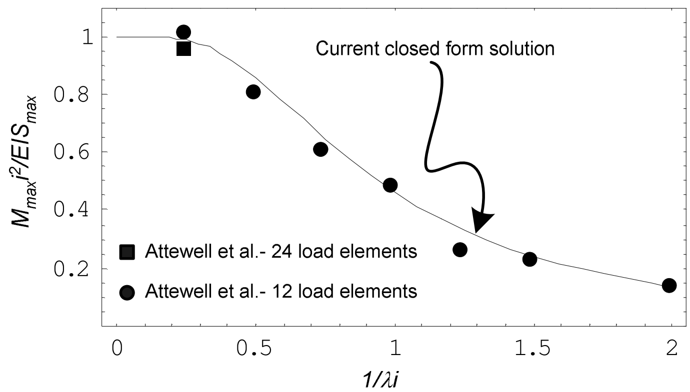


Fig. 6 comparison between closed form solution and previous solution

THE VALIDITY OF VESIC'S SUBGRADE MODULUS FOR PIPE-SOIL-TUNNEL INTERACTION

The solution for the Winkler system (Eqs 15 or 16) requires the knowledge of subgrade modulus *K*. Attewell et al. (1986) suggest the use of the Vesic (1961) equation for the subgrade modulus, which is given by

$$K_{\infty} = 0.65\sqrt[12]{\frac{EsB^4}{EI}}\left[\frac{Es}{1-\nu^2}\right] \qquad \text{Eq. 17}$$

where *B* is the width of a beam (in our case $2r_0$). This equation refers to a beam resting on the surface of an infinite half space.

The physical meaning of this subgrade modulus is as follows. If this subgrade modulus is used to define the maximum moment in an infinite Winkler beam under a concentrated load, the moment is computed as:

$$M = \frac{P}{4\lambda} = 0.37Pb\left(\frac{EI}{Esb^4}\right)^{0.27}(1-\nu^2)^{0.25} \qquad \text{Eq. 18}$$

where $2b=B$.

As a reference, the Biot (1937) solution for the same conditions (i.e. concentrated load on an infinite beam) but for elastic continuum is:

$$M = 0.37Pb\left(\frac{EI}{Esb^4}\right)^{0.277}(1-\nu^2)^{0.277} \qquad \text{Eq.19}$$

The two expressions are practically the same, and Eq. (19) provides the physical meaning of the Vesic (1961) equation which is simply an analog, essentially allowing a beam on

Winkler foundation to exhibit similar displacements and moments to that of a beam on an elastic foundation when loaded with concentrated loads.

Due to the fact that the above analog refers to a beam resting on the surface of an infinite half space, Attewell et al. (1986) suggested taking twice the value of the Vesic expression since the pipe is buried in the soil, $K = 2K_\infty$.

The basis for creation of any analog is that it should have the same input. Vesic (1961) derived Eq 17 as an analog on the basis that the two systems (i.e. Winkler and elastic continuum) are loaded by the same external loads. However, in the case of the effect of tunneling on existing pipelines the basis for creating an analog should be an identical input of green field settlement profile. As shown in Fig. 5, the tunnel effect may be represented by a force distribution along the pipe which relates to the green field settlement. Only if this force distribution in both systems (i.e. Winkler and continuum) is equal, the use of Vesic’s expression will result in identical bending moments in the two systems. This force distribution, *f(x)*, at the level of the pipe, is equal to that which will cause a green field settlement in a pipe-less system. In a Winkler system this force distribution is equal *f(x)=KSv*, while in the continuum solution, presented in matrix form, it is $\{f\} = [\lambda s]^{-1}\{Sv\}$, where λs is the flexibility matrix of the soil $[\lambda s]_{i,j} = G_{i,j}$ (note, this is different than $[\lambda^* s]$). These two force distributions are not generally the same, and hence the Vesic expression might not necessarily be adequate for the current problem.

For comparison purposes two non-dimensional controlling parameters (EI/Esr_0^4 and i/r_0) are considered and varied. In the elastic continuum solution $R = (r_0/i)^3 EI/Esr_0^4$ and is a function of these two parameters. For the Winkler closed form solution, it can be shown that when using $K = 2K_\infty$ and ν=0.25, $\lambda i = 0.813\left(EI/Esr_0^4\right)^{-0.27}(i/r_0)$, which can then be substituted into Eqs (15) or (16) to compute the maximum bending moment. Fig. 7 shows the comparison between the normalized bending moment resulting from the continuum elastic analysis and the Winkler solution using Vesic’s expression.

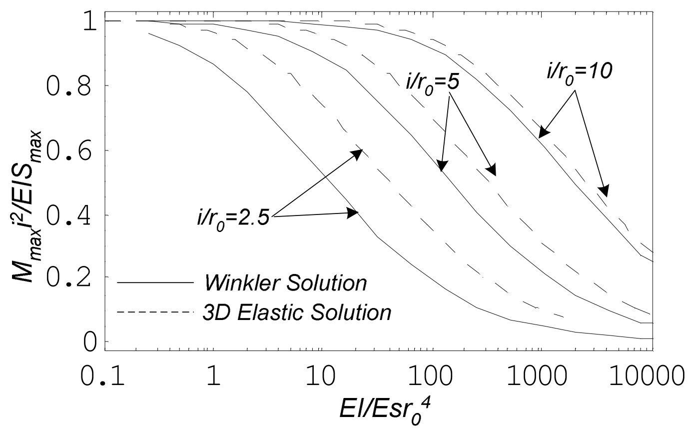


Fig. 7 Comparison between continuum solution and Winkler solution using the Vesic analog

For i/r_0=10 the agreement between the two solutions is quite good, and from a practical point of view they are identical. However, as i/r_0 decreases, the difference increases, resulting in significant differences in values. This suggests that the Vesic analog is not necessarily adequate for all cases, and if the soil is assumed to be linear elastic, the Winkler solution may not be conservative (i.e. underestimation of bending moments). In the following section a revised subgrade reaction modulus for use in the Winkler system is suggested based on the analogy of identical green field settlement input.

AN ALTERNATIVE ANALOG FOR WINKLER SOLUTION

Since the maximum bending moment is often a parameter that controls the possible pipe damage due to tunneling underneath, it was chosen as the entity for the comparison between the Winkler and the elastic continuum systems. A subgrade reaction modulus that will result in similar bending moments in the Winkler and the elastic continuum systems is proposed here.

In the Winkler system the normalized bending moment is a function of λi as shown in Fig. 6, whereas in the elastic continuum it was found to be a function of R. Strictly speaking, there is an influence of depth Z/r_0, but it is relatively small as described previously. It was found that in order for the functions to fit closely, $\lambda i \cong \sqrt[4]{3/R}$. Rearrangement of this equation leads the subgrade modulus to be equal to:

$$K = \frac{12Esr_0}{i} \qquad \text{Eq. 20}$$

The coefficient of subgrade reaction, defined as $k=K/D$ (where D=pipe diameter), is therefore equal to $k=6Es/i$. Fig. 8 shows the comparison between the Winkler solution with the subgrade modulus of Eq. 20 and the continuum solution as before. Good agreement exists between the two. Hence, it is proposed to use Eq. 15 or 16 with Eq. 20 to compute the maximum sagging bending moment of a pipe subjected to tunneling underneath. It should however be noted that the current analysis is based on an assumption that the soil is linear elastic. In reality soil nonlinearity will be involved and this requires further investigation. Nevertheless, for small displacement where elastic behavior prevails the elastic continuum solution is still valid.

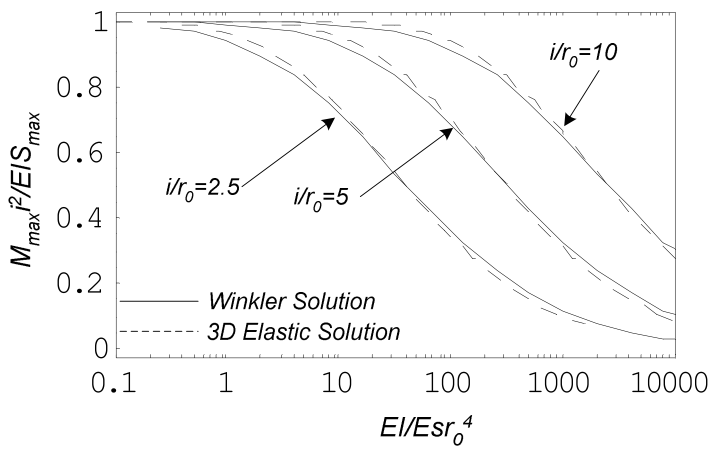


Fig. 8 Comparison between continuum solution and Winkler solution under the new analog

CONCLUSIONS

A problem of tunneling effects on existing pipelines was solved using a rigorous elastic continuum method using Mindlin's Green function and a more simplistic Winkler system. The elastic continuum solution was compared to the Winkler solution with a subgrade modulus based on Vesic equation, which was employed by Attewell et al. (1986). It was found that Vesic's expression, which was originally derived to give the same moments and displacements under a concentrated load in a Winkler system as in elastic continuum, is not necessarily adequate for the problem of tunneling effects on pipelines and may not be conservative. An alternative expression for the subgrade modulus is proposed for Winkler system and this gives the maximum bending moments similar to the elastic continuum systems.

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