

BEHAVIOUR OF A SOIL PASTE CONTINUUM

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ABSTRACT: Many geotechnical engineers who apply Coulomb's equation (1) in plastic limit calculations are not aware that Coulomb stated that **newly disturbed soil has no cohesion**. Values of cohesion c and friction ϕ used by geotechnical engineers for undisturbed soil or excavated and re-compacted soil should reflect the contribution of Taylor's interlocking to peak strength. The lack of cohesion in excavated and re-compacted soil increases the importance of ductility in soil. Soil should not be compacted too much below critical state density. Over compaction to obtain higher stiffness and strength increases the risks associated with cracking and seepage.

1 COULOMB'S EQUATION

Terzaghi quoted Coulomb's Essai (received 1773, published 1776) as the source of "Coulomb's equation" (1) for the peak strength of soil

$$\pm \tau = (c + \sigma' \tan \phi) \quad (1)$$

This paper will begin with points made by Schofield (1998) about Coulomb's Essai.

1.1 Dilatation on Couplet's slip plane.).

Heyman's (1972) English translation of Coulomb's Essai, with his own essay on Coulomb's civil engineering contributions, is a source on which these subsequent English language studies are based. Heyman copies old figures that show stacks of cannon balls, and sections of fortress walls, which were familiar sights in Coulomb's time, and reports ideas dating from 1699. Discrete planes of slip are seen through ground behind the walls that fail. Stacks of cannon balls have cleavage or slip planes. These failure mechanisms provided Coulomb and his contemporaries with models on which they based their concepts. Couplet (1726), (Heyman's Fig. 5.5), shows a close packed stack of spheres with a slip plane LK. It slips in the direction of the tangent plane slope LB at contact of spheres. Introducing v as the angle BLK between the slip direction LK, and the slip plane direction LB, then v is the angle of what is called interlocking or dilatation. But 47 years later, Coulomb's Essai missed Couplet's point. Couplet had already shown motion both normal and parallel to a slip plane, but Coulomb's compression failure on an inclined slip plane only has motion in the direction of the slip plane CM (Fig 5 from Coulomb's Plate 1).

1.2 Strength on slip planes in Coulomb's Essai

Coulomb considered ground as a body of homogeneous material with two possible mechanisms of failure into separate parts either (i) by slip on a sliding surface with $v = 0$, or (ii) by cleavage on a fracture surface. A vector of effective stress across a slip surface or cleavage has two components; σ' is effective stress normal to the plane, and τ is shear stress in the slip plane direction. Terzaghi

taught that, in water saturated soil, total stress $\sigma = (\sigma' + u)$ consists of two parts. One part is the pore water pressure u , which has no measurable effect on strength; and only the other part, the effective stress σ' can produce changes in compaction and soil strength.

The concept in Equation (1) that peak strength has two parts, cohesion and friction, predates Coulomb. His *Essai* begins by stating the teaching at his engineering school at Mézières; resistance to slip combines a "friction" component, with a "cohesion or adhesion" component. Cohesion in shear is supposed to be equal to adhesion in tension, and to be independent of the force acting normal to the plane of slip, and proportional to the area of sliding contact. "Friction" is supposed to be independent of the area of sliding contact and proportional to the force acting normal to the plane of slip. Equation (1) gives two alternative values of $\pm \tau$ for the possible \pm directions of slip. Hence Coulomb could calculate two limiting lateral forces A and A' of ground behind a retaining wall CB in Fig 8, Plate 1. These "maximum and minimum" values were later called "passive" and "active" forces by Rankine. Coulomb (as translated by Heyman 1999) writes that

"the difference between forces in fluids for which friction and cohesion are zero and those for which these quantities cannot be neglected is that for the former, the side CB of the vessel containing them can be supported only by a unique force, while for the latter there is an infinite number of forces lying between A and A' which will not disturb equilibrium."

1.3 Soil as a continuum

Coulomb and Rankine supposed that slip planes have no thickness. They have a concept of two directions of slip on the failure plane, but do not discuss the magnitude of slip displacement or of shear strain or volume strain. Their slip-plane concepts were the basis of Terzaghi's research and teaching in soil mechanics and of the shear tests of Hvorslev (1937). Roscoe (1953) devised a new simple shear apparatus SSA in which a small slip plane area could experience simple shear strain. In effect it was a shear box with rotating ends, on each of which there was a piece of lubricated rubber sheet. The SSA was flawed from the outset because the lubrication meant that when the ends were at right angles to the sliding surface there was no complimentary shear stress on them, so the specimen did not experience pure shear stress. Roscoe argued that the triaxial apparatus was flawed because displacements were controlled at the ends of the test cylinder and stress was controlled round the cylindrical rubber sheath, but the SSA data proved less useful than the Imperial College triaxial test data. Schofield and Wroth (1968), in *Critical State Soil Mechanics (CSSM)*, discarded thin slip planes and followed principles of continuum mechanics. We adopted a homogeneous continuum, with no particular potential planes of slip or cleavage. Our continuum consists of many small finite elementary volumes, each filled with an aggregate of grains of soil with a great diversity of shapes and sizes. Our typical volumetric element is a cylinder such as the soil specimen in the Imperial College triaxial tests. Plastic yielding flow causes axial compression and radial expansion of such cylindrical elements.

A slip plane is not the cause of failure; it is the consequence of localization of dilatancy in a dense aggregate. As instability propagates through the continuum it leaves a layer of loosened grains as gouge material in critical states. They are the same states that are found in a loose heap of aggregate standing at an angle of repose. In Coulomb's time, internal friction of soil was expressed as the angle of repose ϕ_{crit} of a natural slope. The sand in an hourglass is selected to have hard rough irregular grains that flow again and again without damage to grains. In an hourglass a trickle of sand falls on to the top of the heap in the lower half of the glass. A little dense sand cone with slopes slightly steeper than ϕ_{crit} can be observed at the tip of this heap. This dense cone fails, slumping first in one direction and then another, sending small quantities of loose sand down the slope in all directions. It is impossible to climb up a slope at an angle of repose. Consider a foot walking up the slope. At right angles to the sole of the foot, consider a cylindrical volume element of soil in the slope at repose with axes inclined at an angle $(90 + \phi_{crit})/2$ to the horizontal. As the foot applies an increment of load this cylinder will yield. The cylinder of loose soil experiences axial shortening and radial expansion. The foot moves down but the walker does not move up the slope.

Soil selected for construction is an aggregate of rough irregular hard grains that flow when loose and become strong when densely compacted. Soil mechanics should discuss how compaction increases strength. Selected soil as used in construction of roads, embankments and levees (and aggregate selected for concrete) is stockpiled in loose heaps, which stand at an angle of repose. The grains must not be friable (not crush). Loose soil flows in front of a grader blade and fills holes to level an irregular surface, but when loose aggregate is compacted in place it forms a strongly interlocked dense and stiff solid. There must be no change of grading when dense soil is re-excavated. Useless friable soil like broken shale or chalk is discarded in spoil heaps. In farming, land is ploughed to loosen the grains and assist penetration of roots. Soil must become firm when pressed down round the roots of a plant, without the grains being broken down to an impermeable dust. Useful soil has strong solid grains. Presumably Coulomb selected such soil for construction of fortifications and it was generally well compacted with heavy rams. An inspector's cane or walking stick should only penetrate to a small depth in compacted soil. Coulomb used equation (1) for all fortress construction materials, including brick and rock, natural and made ground.

1.4 The effect of compaction

The question of compaction is addressed in Couplet's concept of slip surfaces that separate, and in calculations of the work done to change the volume of soil in the slip zone. In the field when dense fine grained soil such as stiff clay fails there is both volume and water content increase in the disturbed soil paste on the slip surface, called "gouge material". Slip surfaces in London Clay appear wet and slick and are called "slickensides"; tunnel failures in London Clay are called "greasy-backed". Increased water content and softening is linked with the change of packing density of this disturbed fine-grained London clay soil. It is equally evident in failure of dense coarse sand. Dark lines on X-ray plates of a body of dense sand indicate failure planes where there is loosening of the dense coarse sand in the gouge material on the slip planes. In either case, with fine or coarse grain soil, a volume increase against normal pressure requires additional work to be done equal to the volume increase times the pressure acting normal to the slip plane.

In section 14.9 of his textbook, Taylor (1948) described a drained shear box test on dense sand under a normal load of 3 tons per sq. ft. The peak strength was 1.94 tons per sq. ft. At that point the sample thickness was increasing with 0.0017 in. of thickness increase per 0.01 in. of shearing displacement. Taylor calculated a sliding friction coefficient $1.94 / 3 = 0.645$. A part $0.0017 / 0.01 = 0.17$ he called interlocking. The remaining portion $0.645 - 0.17 = 0.475$ he attributed to work dissipated in friction. Interlocking motion, which is dilation at an angle ν , adds an additional strength that can be taken into account by adding this dilation angle ν to the angle of friction ϕ_{crit} . When friction and interlocking are taken together the peak strength can be expressed as an apparent coefficient of sliding friction

$$\mu = \tan \phi = \tan(\phi_{crit} + \nu) \quad (2)$$

The phenomenon of dilation occurs in every strength test of dense soil whether it is undisturbed or excavated and newly compacted, and is the cause of part or all of any "apparent cohesion". The data of peak strengths of over-consolidated or compact soil may be expressed either (i) by fitting all data points to one line with equation (1) with an apparent cohesion c and friction ϕ , or (ii) by fitting each point with an appropriate value of $(\phi_{crit} + \nu)$. The latter choice is consistent with Coulomb's law for newly disturbed and newly compacted soil.

2 COULOMB'S LAW

Students of Terzaghi's soil mechanics will not find statements that they may expect to find in Coulomb's writing. They will find statements they may not expect.

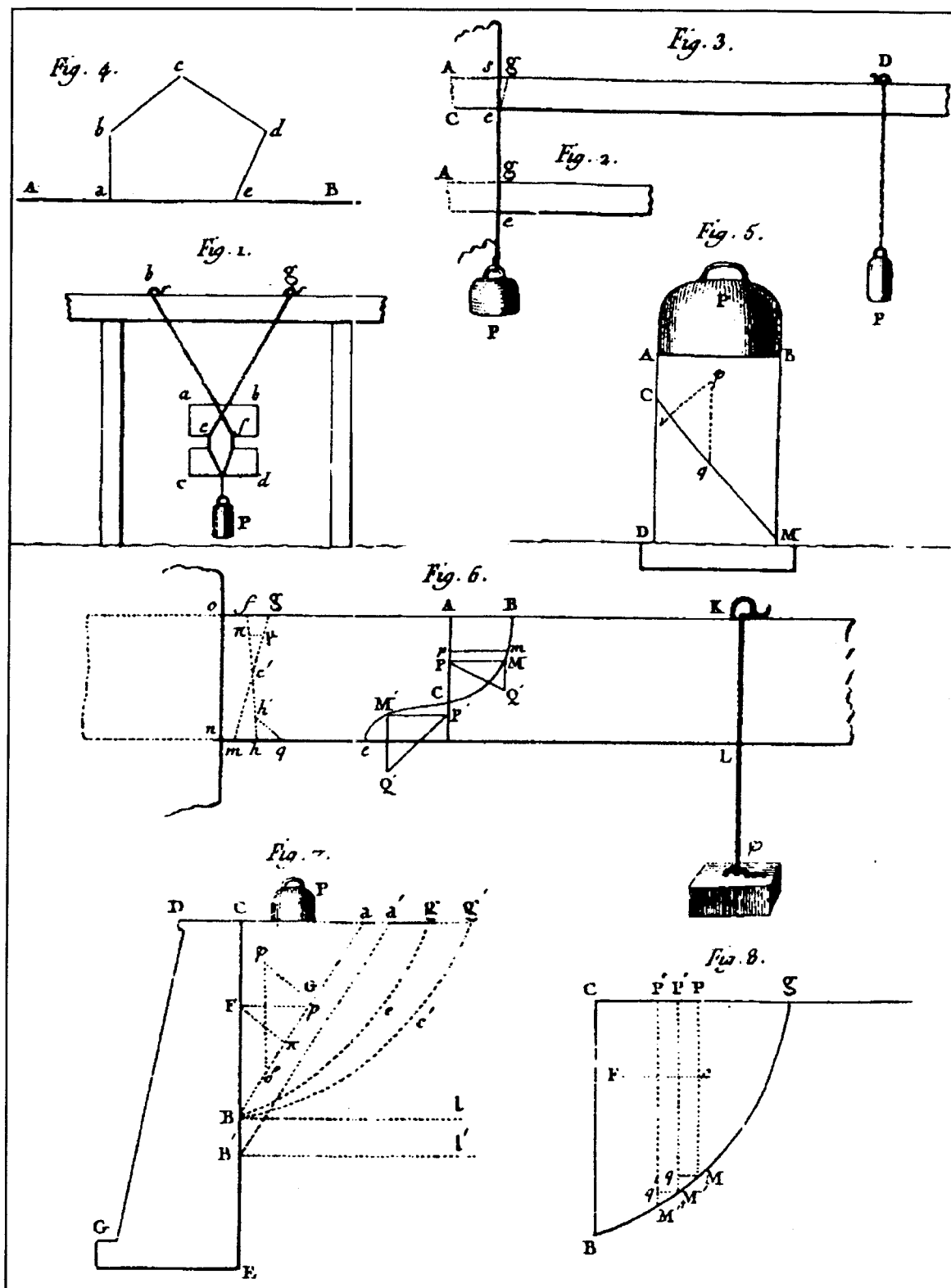


Plate 1. After Coulomb (1773).

2.1 Leslie's criticism of "roughness" theory

There is evidence in Leslie (1804) linking Coulomb with the mistaken concept that friction is dilation equation (3), although Coulomb (1773, 1781) does not appear to have written it down,

$$\phi = v. \quad (3)$$

John Leslie (1766-1832) was a Scottish scientist, elected to be Professor of Physics in the University of Edinburgh because of his work on radiant heat in a book, Leslie (1804), in which he also discusses the generation of heat by friction. Bowden and Tabor (1973), in the 1st edition of their study of sliding friction, quote part of Leslie's book (pp 299-305). Leslie writes about frictional resistance to sliding as follows.

"If the two surfaces which rub against each other are rough and uneven, there is a necessary waste of force, occasioned by the grinding and abrasion of their prominences. But friction subsists after the contiguous surfaces are worked down as regular and smooth as possible. In fact, the most elaborate polish can operate no other change than to diminish the size of the natural asperities. The surface of a body, being moulded by its internal structure, must evidently be furrowed, or toothed, or serrated. Friction is, therefore, commonly explained on the principle of the inclined plane, from the effort required to make the incumbent weight mount over a succession of eminences. But this explication, however currently repeated, is quite insufficient. The mass which is drawn along is not continually ascending: it must alternately rise and fall: for each superficial prominence will have a corresponding cavity; and since the boundary of contact is supposed to be horizontal, the total elevations will be equaled by their collateral depressions."

Bowden and Tabor show a saw tooth slip plane with slopes up to the tips of successive teeth, and they write that Leslie's criticism of "roughness" theory

"remains unanswered. Of course one can say that work is used in dragging a body up a slope and when it gets to the top it falls with a bang, bending and denting the surface, so that all the work done on it is lost as deformation work during impact. If we adopt this view we have gone a long way from the dragging-up-the-roughness model. We are really talking of a deformation mechanism."

Bowden and Tabor go on to discuss the welding of small areas of contact between metal sliding surfaces, on which they base a theory of sliding friction at a contact between solids.

2.2 Friction in Coulomb's "Théori des Machines Simples"

Coulomb (1781) won a double prize of the French Academy for the paper on friction on bearing surfaces that included the "friction circle" analysis for plane journal bearings in simple machines. It also included slip-ways for launching ship's hulks. He experimented on surfaces of all materials with all lubricants then available to Naval architects including various types of wood and grease. That paper speculates that when wood slides on wood, wood fibers brush against each other. As fibers slowly bend they store energy and, when an end of a fiber brushes past a restraint and springs free, part of the energy stored in it is dissipated. Bowden and Tabor show Coulomb's illustration of these brushes of wood fibers. There are transient flows of lubricant among fibers when they move, or when sliding stops or is restarted. Coulomb observed transient variations of resistance to sliding for wooden surfaces that he did not observe for metal surfaces. Nowhere in that paper does Coulomb write equation (3). Leslie may be quoting another paper, or something that he heard when he visited Europe in Coulomb's lifetime.

2.3 Coulomb's adhesion and cohesion tests.

Coulomb had been taught that adhesion (resistance to tensile separation) is identical to cohesion (resistance to shear on a slip plane). He decided to test if this was true for limestone Figs 1 and 2, Plate 1. He notched a square slab one inch thick with a neck two inches wide and loaded it to tensile failure at cohesive strength of 215 lb./in. He had a short beam loaded by rope that passed round it close to the clamped end so that it had a force directed along the plane of fracture. It failed with cohesion of 220 lb./in. He repeated these tests several times and found that cohesion was nearly always larger than adhesion. Clearly these were not identical physical effects, but the fractional difference of only 1/44 was negligible for practical purposes. So in practice Coulomb took the strength of ground in shear to be the sum of any adhesion (measured as direct tensile strength) plus internal friction (effective normal pressure times the tangent of the angle of repose ϕ_{crit} as observed in natural slopes).

Coulomb was later to make fundamental contributions to physics, so we must take him seriously when he says his *Essai* on civil engineering is of "practical use", but is his "weakest work". He says he wrote his *Essai* for his own use when he was responsible for building fortifications to resist cannon fire, a very practical matter. Coulomb quotes use of a load factor of 1.25 in design of fortress walls. He gives examples of earth pressure calculation at three places in his *Essai*, in each of which he takes $c=0$. In his tests adhesion was equal to cohesion; he naturally assumed that newly disturbed soil filled behind walls had no adhesion and hence in three separate places he states that *newly disturbed soil has no cohesion*. After reading Terzaghi (1943) I did not expect this simple statement by Coulomb, but it seems to me (Schofield 1998) to merit being called "Coulomb's Law" more than the strength equation (1) that is so widely taught. It fits the critical state equations described later. It corrects errors that arise in teaching of Coulomb's equation

3. STRESS AND MOHR'S CIRCLE.

In Coulomb's time the stress in a continuum was not understood. He did not appreciate that the vectors of stress across two planes through a point in a continuum could not be specified independently. He attempts a calculation by the method of slices (Fig 8, Plate 1), supposing that limiting stress acts both on the inclined slip plane and also on the vertical plane between slices. The vector of effective stress across Coulomb's slip plane in equation (1) has components σ' and τ . The vector of displacement can have a component normal to and one parallel to the slip plane. The relationship between these four vector components is not simple. In contrast, continuum mechanics explains the effects of bulk volume changes and shear distortion changes in a continuum.

3.1 *Stress and strain invariant and increment.*

Rankine (1847) taught Lamé's representation of stress components at a point in a continuum (Cauchy 1823) by an ellipse. Culmann and Mohr introduced the representation of stress by a circle (1870). Each point on the ellipse, or circle, represents a vector of stress that acts on one particular plane through a point in the continuum. The word "tensor" comes from the array of numbers needed to define a stress. We first learn about physical quantities such as the temperature of a fluid that can be described by one number at one point. These are scalars, or tensors of zero order. Velocity is a physical quantity that requires several numbers to define it at one point in a fluid. It is a first order tensor (vector) v with one component of velocity v_i, v_j, v_k , in each direction i, j, k . Stress is a second order tensor with a stress vector across each plane through a point at which stress is defined. Once we select the directions of our 3 reference axes we get 3 components of normal and 3 of shear and complimentary shear stress on our 3 orthogonal planes. Although we need 6 numbers to define a second order tensor in 3 dimensional space, stress is only one single physical quantity. The symmetrical array of numbers that define the physical quantity "stress" exhibits invariance under change of reference axes.

The triaxial test specimen under axial stress increment σ'_1 and radial stress increment $\sigma'_r=0$ (see CSSM Fig 2.8) experiences spherical stress increment $p' = \sigma'_1 / 3$, plus two pure shear stress increments also $q = \sigma'_1 / 3$. These cause spherical volumetric compression, plus two pure shear distortion increments. CSSM writes that axial compression is only partly due to spherical compression and mostly caused by shearing distortion. Conversely, indirect swelling is the difference between shearing distortion and spherical compression, and CSSM writes

"Consequently, we must realise that Young's Modulus alone cannot relate the component of a tensor of stress increment that is directed across a cleavage plane with the component of the tensor of compressive strain increment that gives the compression of a fibre embedded along the normal to that cleavage plane. An isotropic elastic body is not capable of reduction to a set of three orthogonal coil springs."

CSSM deliberately did not begin with Mohr's circles and slip surfaces:

"on the grounds that Mohr's representation of stress imparts no understanding of the interrelation of stress-increment and strain-increment in elastic theory, that it plays little part in continuum theories, and that the uncritical use of Mohr's circle by workers in soil mechanics has been a major obstacle to the progress of our subject."

My own development of continuum mechanics for soil, Schofield (1959), introduced scalar invariant quantities to describe states of soil. To represent the elastic test paths of triaxial test specimens under axial stress σ'_a and radial stress σ'_r , I chose the quantities (p', q, v) where v was the volume of space occupied by unit volume of grains later called specific volume. The scalar $p' = (\sigma'_a + 2\sigma'_r)/3$ is related to the first invariant of the effective stress tensor. The deviator stress magnitude $q = (\sigma'_a - \sigma'_r)$ is related to the second invariant of the stress deviator tensor. These parameters define average conditions in a random aggregate of irregular soil grains. In a gas the pressure depends on average velocity of gas molecules in random motion, and the gas laws relate scalars such as density, pressure and temperature. The material constants for an elastic continuum are a bulk modulus K relating change of volume to change of pressure and a shear modulus G relating shear stress to shear strain. CSSM theory applies to a class of soil for which triaxial test data in terms of invariant parameters (p', q, v) fit two "critical state" equations that define the flow of a paste of soil grains and water. An aggregate of grains held together by stress p' initially acts as an elastic body.

3.2 Elastic properties of compact soil paste.

Before it yields and flows, a compact aggregate of hard irregular grains held together by effective stress p' acts as non-linear elastic solid material. Terzaghi and Casagrande used values of v and $\ln p'$ to plot non-linear data of recoverable strain in volumetric compression and swelling of compacted soil aggregate. Data for over consolidated fine-grained soil or dense sand fit a function $p' = p'(v)$. CSSM followed Terzaghi and Casagrande and adopt the function $p'(v) = \exp\{(v_\kappa - v)/\kappa\}$, so

$$v + \kappa \ln p' = v_\kappa, \quad (4)$$

The significance of this Equation (4) was that it already had been used to fit data. An alternative form of Equations (1) to (4) could give dimensionless equations by dividing p' everywhere by the shear modulus of the solid grains. Any function by which increased pressure p' caused decreased compressibility dv/dp' might be tried. But whatever the function, one value of v_κ in which effective pressure p' and specific volume v are combined, defines one aggregate of grains in which strain increments are non linear but reversible.

4. PLASTIC STRAIN OF DENSE AND LOOSE SOIL AGGREGATES

4.1 Reynolds "dilatancy".

Dense soil exhibits what Osbourne Reynolds called "dilatancy". He demonstrated this with two soft rubber bags, one bag filled only with coloured water and another bag densely filled with small lead shot with coloured water in the interstices between the spheres. He connected each bag to a glass tube that served the purpose of being a gauge to measure the dilation. He squeezed each bag in turn. The water in the bag without shot rose in the tube. On the contrary, the water in the bag with the shot sank as water was drawn into the interstices in the dilating aggregate of small spheres. He noted that

"A well-marked phenomenon receives its explanation at once from the existence of dilatancy in sand. When the falling tide leaves the sand firm, as the foot falls on it the sand whitens, or appears momentarily to dry round the foot. When this happens the sand is full of water, the surface of which is kept up to that of the sand by capillary attractions; the pressure of the foot causing dilatation of the sand, more water is required, which has to be obtained either by depressing the level of the surface against the capillary attractions or by drawing water through the interstices of the surrounding sand".

He expected that recognition of this property of dilatancy "will place the theory of earth pressure on a true foundation", but he left earth pressure study to others. He thought the property "places a hitherto unrecognized mechanical contrivance at the command of those who would explain the fundamental arrangement of the universe" and came to believe that "the luminiferous ether" was an aggregate of minute perfectly elastic spheres. Reynolds (1902) recalled Lord Kelvin's theory of vortex atoms which "afforded the first conception of matter passing through a space completely occupied by matter without resistance". That theory had been found impossible, but in its place Reynolds said he had a new theory "with none of the difficulties of its predecessor". He developed his theory in a book "On the Sub-Mechanics of the Universe", which Cambridge University Press published. Professor Jack Allen (1970) quotes Sir J. J. Thompson, (Nobel Laureate, and the most eminent of Reynolds pupils at Manchester University) as frankly describing this work as "the most obscure of all his writings, as at this time his mind was beginning to fail .."

An additional basic problem is representation of dilatant soil as an elastic material. If soil is elastic then a clockwise shear stress increment must produce a clockwise elastic strain increment. If clockwise shear strain causes volume increase then anti-clockwise shear strain must cause volume decrease. This can be seen in a saw tooth slip plane where movement to the right causes expansion and movement to the left causes compression. Hence elastic shear strain may not cause any volume change because if an elastic soil had this property there would have to be two classes, one of clockwise and another of anti-clockwise elastic soils. Introduction of plastic theory introduces both stress and stress increment tensors, both elastic and plastic strain increment tensors, and solves the problem. For both clockwise and anti-clockwise shear strain increments, plastic volume changes found by the associated flow rule of the theory of plasticity are positive (dilation) if the effective spherical stress p' is less than critical and are negative (contraction) if it is greater than critical.

4.2 Casagrande's critical voids ratio and Cambridge critical states.

Reynolds' tests are quoted in the original paper on liquefaction Casagrande (1936). He describes drained shearing tests in which dense sand expanded, very loose sand reduced its volume, and in between these there is a critical voids ratio (CVR) at which drained shear takes place at constant volume. Casagrande's original CVR was independent of pressure. He supposed liquefaction was flow of sand in states more loose than the CVR and he proposed a method of predicting pore pressures in an undrained shear test of such loose sand. Schofield and Togrol showed that Casagrande's method contained an error. It supposed that soil at ultimate effective pressures in undrained tests will "remember" the initial states at which the tests begin. A different concept of critical states was proposed by Roscoe Schofield and Wroth (1958) based on analysis of paths of triaxial tests of both fine-grained clayey silt and coarse grained sand. Test paths begin in elastic states; soil yields with large shear strain increments, and flows in critical states with no memory of any initial state. In CSSM the elastic states are given by equation (4) and the critical states satisfy equations

$$q = M p' \quad (5)$$

and

$$v + \lambda \ln p = v_\lambda = \Gamma \quad (6).$$

Roscoe and Schofield (1963) Plate 2 Fig 4, superposed the critical state curve EE'' with equation (6) across elastic compression and swelling curves EC and $E''C'$ each with different values of v_κ in Equation (4). To the right of EE' are initial states with $v_\lambda > \Gamma$ (more wet than critical) where soil paste yields in a ductile plastic manner with plastic compression during plastic strain. To the left are initial states with $v_\lambda < \Gamma$ (more dry than critical) where there is either (i) plastic dilatation during plastic strain (this dilation is localised in thin layers of gouge material on slip planes) or (ii) tension cracking. These behaviours will be mapped in detail below and in Plate 3.

4.3 Loose heaps and flow of soil paste in critical states.

Returning to the slope at an angle of repose, the fact that soil flows at the surface shows that soil there is at a critical state. Below the surface the effective pressure increases, which might tend to reduce the strength in equation (2). But if it were reduced the slope would fail at depth. The fact that the heap retains the same angle of repose regardless of the height of the heap shows that the internal friction of the soil is constant at all pressures, as stated in equation (5). So a granular medium that exhibits a constant angle of repose also satisfy the critical state equations. All volume elements at all depths below a slope at repose are in critical states. In CSSM we did not attempt to analyze the micro mechanics of granular flow, and we wrote about internal friction as follows

"Consider a random aggregate of irregular "solid" particles of diverse sizes which tear, rub, scratch, chip, and even bounce against each other during the process of continuous deformation. If the motion were viewed at close range we could see a stochastic process of random movements, but we keep our distance and see a continuous flow. At close range we would expect to find many complicated causes of power dissipation and some damage to particles; however we stand back from the small details and loosely describe the whole process of power dissipation as "friction", neglecting the possibilities of degradation or of orientation of particles."

We felt we did not need to know which factors determine the angle of repose of a loose heap of selected soil or aggregate. CSSM introduced a coefficient of internal friction M to generalize the sliding friction coefficient μ , and found a relationship between M and the angle of repose ϕ_{crit}

$$M = 6 \sin \phi_{crit} / (3 - \sin \phi_{crit}) \quad (\text{for example, if } \phi_{crit} = 30, \sin \phi_{crit} = 1/2, M = 1.2)$$

Roscoe, Schofield, and Wroth (1958) introduced $p' = p'_{crit}$ in equation (6) and found the ultimate effective stress p'_{crit} in soil paste at large plastic strain during large undrained deformation at constant specific volume v as

$$p'_{crit} = p(v) = \exp \{(\Gamma - v)/\lambda\} \quad (7)$$

A sample of soft clay held in the hand has zero total stress. An air-water interface covers every open pore in the surface of the aggregate. There will be suction, a negative pore water pressure in every pore, and there will be an equal and opposite positive spherical effective pressure p' everywhere in the soil aggregate. The suction can be observed if the sample is placed on a suction plate. In rapid undrained frictional flow a soil paste such as soft clayey silt is a perfectly plastic material with constant undrained shear strength. The apparent cohesion c_u is suction times friction which is

$$q_{crit} = q(v) = c_u = M \exp \{(\Gamma - v)/\lambda\}, \quad (8).$$

Equations (4), (5), (6) and (8) were developed to interpret shear test data. They can be regarded as "paste laws" for a paste of soil grains and water, in the same way that gas laws define gas behaviour. Strength is a sum of the internal friction and the dilatation in interlocked aggregates when sheared. Differences between coarse sand and fine-grained clayey soil are due to drainage. An aggregate of rough irregular hard grains has large shear strains as it flows down a slope at an angle of repose. It has small increments of plastic compression as the heap rises. The difference between sand and clay soil is the ease of drainage of whatever fluid is in the pores between grains. Both gouge material and granular material at an angle of repose are in critical states. It is a fact of experience that when saturated soil is worked it flows as a plastic paste while being formed into shape. The Greek for forming clay into a shape by hand is $\pi\lambda\alpha\sigma\sigma\epsilon\nu$ ("plassin" to mould) so by definition yielding soil is a plastic material. Classical plasticity describes metal or soft clay flowing with constant volume. In CSSM a new theory of plastic yielding of soil with plastic volume change is developed.

5 THE YIELDING OF SOIL IN STATES NEAR CRITICAL.

5.1 *Plastic yielding without cohesion*

Roscoe, Schofield, and Wroth tried to correlate data in drained and undrained tests by subtracting a "boundary energy correction" from the peak strengths to give a value for internal dissipation. I later proposed that our calculation should be modified by adding an "elastic energy correction" to allow for the release of elastic energy by soil grains in tests where spherical effective pressure p' decreases or the uptake of elastic energy where p' increases. Thurairajah (1961) undertook this work. His analysis had a remarkable conclusion. In a range of drained and undrained triaxial tests with and without volume change the dissipation of work W during increments of distortion ϵ was everywhere the same as at the critical states

$$dW / d\epsilon = Mp' \quad (9)$$

We had speculated in 1958 that additional internal work might be absorbed by adhesion if the average distance between the centers of grains increased. Thurairajah had showed this not to be true. Thurairajah's conclusion is a more general and more powerful form of the work of Taylor.

The triaxial tests from which Thurairajah got data were timed to achieve pore pressure equilibrium by the end of a test and there were questions about the early part of tests. Roscoe wanted to check them with data from his own SSA, but it was evident that the SSA could not impose uniform states of stress and strain because the rotating ends of the box did not apply any shear stress. Caladine (1963) then contributed a crucial idea. Since normally consolidated clay seems to be a strain-hardening plastic material he suggested that it should have a yield locus satisfying the associated flow rule of the theory of plasticity. Data suggested that the yield locus was elliptical and if that were true, plastic strain rates could be deduced. Those data were questionable and it would be several years before we could undertake new tests very much more slowly. It occurred to me that the associated flow rule and Thurairajah's dissipation function would define a yield function. This led to the development of a constitutive model for an idealised soil on the wet side of critical states first called "Wet-Clay", Roscoe and Schofield (1963), then called Cam-clay, Schofield and Togrol (1966) and recently called "original" Cam-clay by Wood (1990).

5.2 *Original Cam-clay.*

A soil paste with a given value of v_k in Equation 4 can be compressed to values $v_\lambda > \Gamma$. As effective pressure increases from p_{crit}' to p' on ABC in Plate 2 Fig 3 (after Roscoe and Schofield 1963) the strength q is seen to fall from AE to BD to zero at C, where CDE has equation

$$(q/Mp') = 1 - \ln(p'/p'_{crit}) < 1 \quad (10)$$

For $(\Gamma + \lambda - \kappa) > v_\lambda > \Gamma$ Equation (11) defines a yield locus such as CBE above each elastic compression curve such as ABC and A'B'C'. Anisotropic compression DD'' with $(q/p') = \eta = (\text{const.})$ follows a line $v_\lambda = (\text{const.}) = \Gamma + (\lambda - \kappa) (1 - \eta/M)$. Isotropic compression follows a line CC'' with

$$\ln(p'/p'_{crit}) = 1, \text{ so that } (p'/p'_{crit}) = e = 2.73. \quad (11)$$

Soil paste with critical states given by Equations 5 and 6 is ductile stable plastic material. The elastic energy stored in the aggregate when subject to compression into elastic states defined by Equation 4 may or may not cause quite large shear strains, but if the soil paste does not exhibit isotropic compression when Equation (11) is satisfied it will be unstable. This is a remarkable prediction. Not only did the equations predict the plastic compression that had been found by Terzaghi, they also suggested an instability of slightly cemented or bonded fine silt that matched the well known behaviour of quick clay. The undrained test path CD'E'' in Plate 2 was predicted and proved to fit slow undrained triaxial test data. The soil paste yields with positive pore pressures. It does not flow with the "chain reaction" among grains that Casagrande expected in liquefaction.

Further analyses flowed from the original Cam-clay model, and were the basis of teaching set out by Schofield (1966) (see my home page <http://www-civ.eng.cam.ac.uk/geotech.htm>) and in CSSM. The undrained test path CD'E'' from the point C in Plate 2 Fig. 3 to the critical state at E'' is close to the "American working hypothesis". The path predicted by Skempton's pore pressure parameters is the straight line CE''. I appreciated that the corner C on the yield locus meant that soil paste in isotropic compression could exhibit shear distortion to such an extent that one dimensional plastic compression had a coefficient of lateral earth pressure $K_0=1$. This led later students to modify the yield curve to an ellipse with a tangent at C at right angles to the plane $q=0$. I did not consider this justified. The ellipse implied that soil paste at C does not yield under a small increment of deviator stress. Cam-clay yielding in isotropic compression at C would yield under any slight stress increment. Early test data had misled us into believing that the yield locus might be an ellipse. Tests reported in CSSM and Roscoe, Schofield, and Thurairajah (1963) that were slower than and superior to early tests fitted original Cam-clay.

5.3 Liquefaction.

CSSM was not at first understood in North America where the words "critical voids ratio" were familiar in connection with earthquake induced liquefaction. According to Casagrande there was a liquefaction risk when $v_\lambda > \Gamma$, but as explained above Cam-clay in such states is a stable ductile plastic material. According to Seed the risk was associated with $v_\lambda \ll \Gamma$ and $p'=0$. In order to get experience of strong ground motion in Cambridge University we developed geotechnical centrifuge earthquake model testing. CSSM gave confidence that soil models prepared with appropriate stress paths would exhibit the major features of the mechanisms of behaviour observed in the field.

Geotechnical centrifuge development was funded by model tests for the construction industry. One such test series concerned the crevasse-liquefaction failure of levees in the lower Mississippi valley when the River is at flood stage. Another concerned seepage flow through a cracked body of Teton Dam core material. Such behaviour could not be studied at full scale by the observational method. In these and many other model tests I observed continuous bodies made of soil paste under different conditions. The behaviour observed in models led to a Map (Plate 3 is Fig 31 from Schofield 1980), with which I will conclude this paper. The idea for the Map follows Skempton

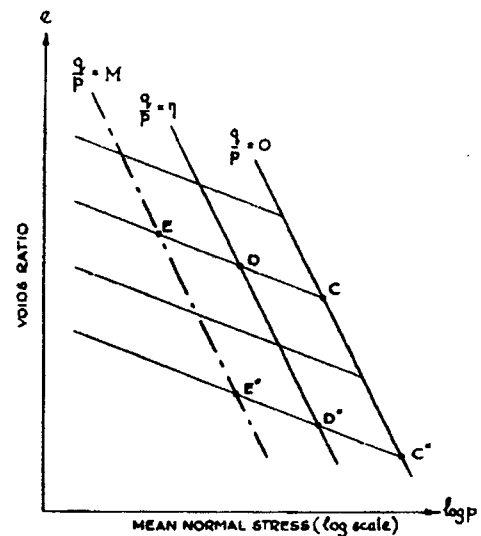
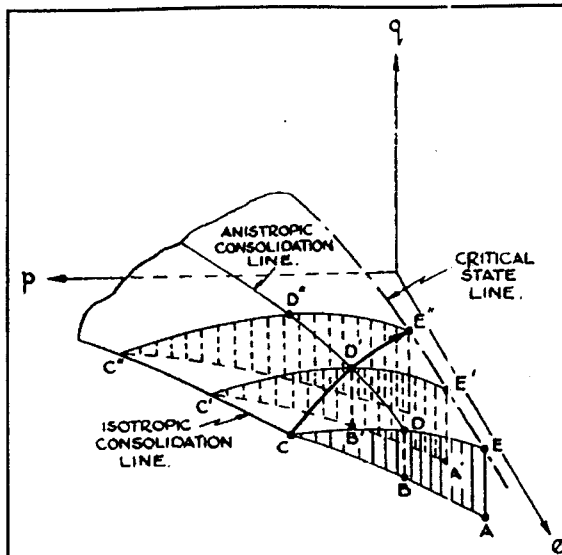


Plate 2. After Roscoe & Schofield (1963).

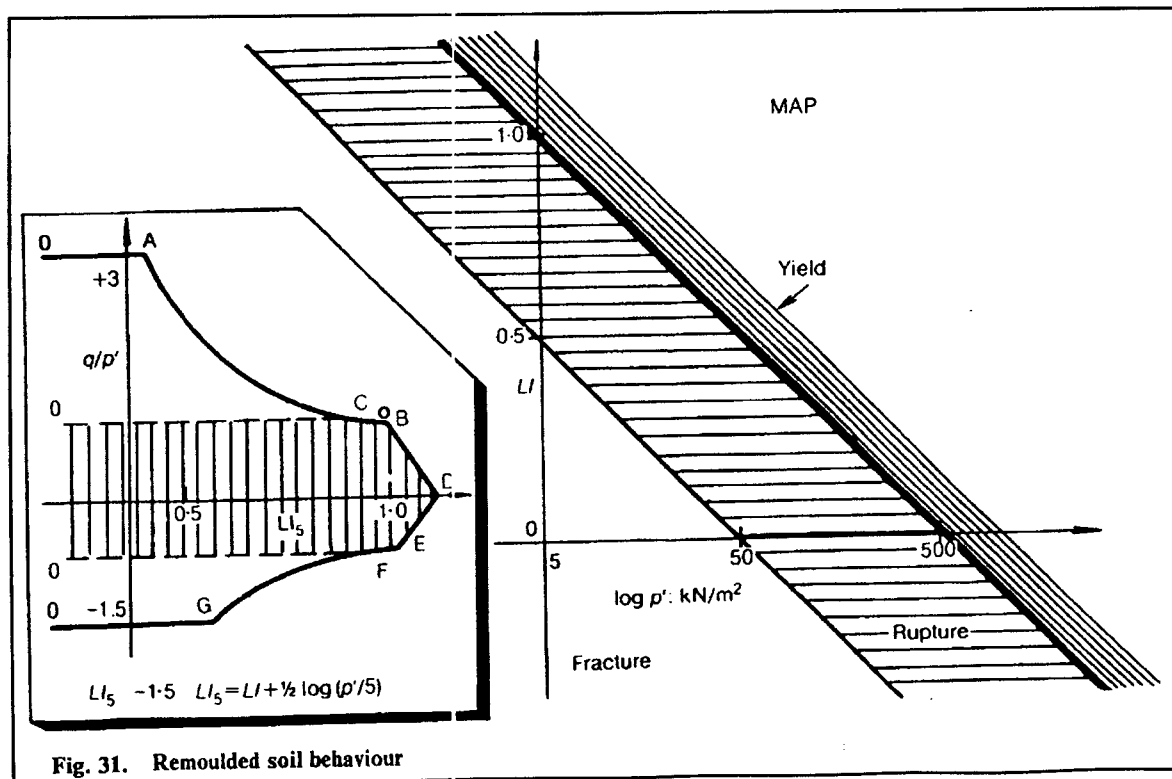


Plate 3. After Schofield (1980).

and Northey (1953) who showed that that in the range of Liquidity index from 1 to 0 undrained shear strength of soft clay increases $(0.1 \text{ psi}) < c_u < (10 \text{ psi})$. Plate 3 shows the Map of Liquidity index and $\log p' \text{ kN/m}^2$. The bold line from $LI=1$ to $p' = 500 \text{ psi}$ is the critical state line. There is an increase of undrained strength by a factor of 100 between the liquid and the plastic limit, therefore p' increases by the same factor along this line. To the right of the critical state line the area shaded diagonally with $\Gamma < v < (\Gamma + \lambda - \kappa)$ is where original Cam-clay applies. To the left of Plate 3 is a section by the plane $p' = (\text{const})$ of the original Cam-clay surface from Plate 2. Where the vertical axis in Plate 2 would show q , the section in Plate 3 shows q/p' . In Plate 2 the horizontal axis would show the voids ratio e . In Plate 3 it shows liquidity index corrected for effective pressure. My correction projects any point on the Map onto the plane $p' = 5 \text{ psi}$. Since $\log 100 = 2$ the correction is

$$LI_5 = LI + (1/2) \log(p'/5) \quad (13).$$

In the section shown to the left in Plate 3 the lines DB and DE show that LI_5 falls linearly with increase of stress ratio $q/p' < M$, as is predicted for Original Cam-clay, and as was found by Shibata. The curves BA and FG show stress ratios at rupture predicted by Equation (1). This is the region of ruptures on thin slip surfaces. One or other of the principal effective stresses in the aggregate becomes zero and soil paste can be expected to show tensile cracking at A where $q/p' = (+3)$, and at G where $q/p' = (-1.5)$. The boundary between the area shaded horizontally in the Map, to which Equation (1) applies, and the un-shaded area where tensile cracking is expected, is uncertain because of the different values of LI_5 at A and at G.

Schofield (1980) described model tests of undisturbed Canadian quick clay and wrote:

"... at low equivalent liquidities, below 0.5, at the moment that soil crumbles or becomes fissured it not only dilates as a mass of blocks but is also rapidly suffused with water. It is the suddenness of this change in permeability that transforms the material from a strong intact mass into a flowslide problem. ... In such cases there are two features of the high risk environment. The first feature is an apparent solidity or strength, which tempts people to build levees on an apparently firm overburden layer .. (which) ... they would never trust if the underlying soft sand was exposed ... (It) tempts people to build housing estates on apparently solid silty (quick) clay which is slightly cemented or bonded they would never trust (it) if they remolded the clay and saw how much water it contained. That is the first feature of the high risk environment. The second feature is the rapid transformation of this apparently solid ground into a soil avalanche: this occurs much more rapidly than would be calculated from diffusion through pores, because of the change of permeability when pores open up...."

Schofield (1982) suggested that liquefaction is one of a range of related sudden events such as piping, or channeling, or boiling, or debris flow, when a body of dense soil near zero effective stress experiences the opening of pore spaces between soil grains while under a high hydraulic gradient. The dam failures in the Baldwin Hills and in Teton fit this pattern. Teton Dam was in a series of dams in which U.S. Bur. Rec. achieved increasing control of compaction and more rapid reservoir filling; the core material was at about $LI_5=0$. In the Baldwin Hills and at Teton the great stiffness and strength that was achieved was evident in nearly vertical faces of each breach. The dams were cracked from top to bottom. The crack faces did not fall even when all the water poured past. Downstream filter layers should have been used, with less heavily compacted material. In centrifuge model tests with Teton Dam core material small cracks were healed as soft soil fell from crack faces, with upward migration of voids. Schofield (1980) and Plate 3 indicates that the core at any given depth should have been compacted only until soil had an equivalent liquidity $LI_5 > 0.5$ at the working effective stress. Teton dam should also have been filled more slowly.

Fort Peck dam was built with a hydraulic fill core placed from pipes supported on wooden piles. Jetting pressures that were used to aid pile driving helped to maintain high pore pressures in the core. This pore pressure propagated below the faces of the dam. Water welled up from relief wells at the down stream toe. Uplift pressures below the flooded buoyant upstream slope reduced effective stress and friction strength. The lateral pressure of the core pushed a short length of the upstream face aside "like a gate swinging open" and hydraulic fill flowed upstream. The soil in the

breach at Fort Peck Dam was at zero effective stress and failed under a high hydraulic gradient; the hydraulic fill flowed out.

These events and Plate 3 show that Seed's concept of a risk associated with $v_\lambda \ll \Gamma$ and $p' = 0$ is consistent with CSSM. There was no liquefaction as reported by Casagrande at Fort Peck, with a "phase transformation" to a "flow structure" on the wet side of critical states; the flow sliding was not rapid transmission by a "chain reaction". Centrifuge tests of compacted bodies of soil paste have helped to explain the nature and risk of liquefaction.

6. CONCLUSION

In the introduction to CSSM we wrote

"We wish to emphasize that much of what we are going to write is already incorporated by engineers in their present judgements. The new conceptual models incorporate both the standard Coulomb model and the variations which are commonly considered in practice: the words cohesion and friction, compressibility and consolidation, drained and undrained will be used here as in practice. What is new is the inter-relation of concepts, the capacity to create new types of calculation, and the unification of the bases for judgement."

It is now clear that the Cam-clay model had more radical consequences than we then supposed. The fact that it closely predicted the data of high-quality, slowly-performed triaxial tests and fitted Skempton's pore pressure parameters validated Thurairajah's conclusion that energy is dissipated in soft clay by internal friction in granular material. Cam-clay confirmed Coulomb's Law; **there is no cohesion in newly remolded soil**. The slope at repose and the fall cone determination of the liquid limit are the phenomena that define internal friction and critical states. The slip plane has a role in the upper bound calculation of theory of plasticity but continuum mechanics and micro mechanics define soil constitutive behaviour. "Mohr's hypothesis" has proved to be inferior to Coulomb's Law and original Cam-clay for newly disturbed soil; the old hypothesis is hardly likely go on being acceptable for undisturbed soil and soft rock. Failures observed in the field and in centrifuge tests in laboratory conditions have emphasized the value of ductility in soil. Plastic design is appropriate to mild steel structures. Soil structures should be well drained and compacted so they operate with $0.5 < LI_5 < 1$ and with $q/p' < M$. The simple plastic design proposed in CSSM is appropriate to a newly disturbed and re-compacted soil paste continuum in such states with internal friction ϕ_{crit} .

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Titles for the three plates are as follows;

Ans-1.jpg Plate 1 after Coulomb (1773),

Ans-2.jpg Plate 2 after Roscoe and Schofield (1963),

Ans-3.jpg Plate 3 after Schofield (1980).

Allen J. & others (1970) *Osborne Reynolds and engineering science today* (edited by McDowell & Jackson). Manchester University Press.