THE “MOHR-COULOMB” ERROR

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Presented on 19 May 1998 in Paris in the Senat at the scientific jubilee of Pierre Habib and to be published in “Mechanique et Geotechnique”, Balkema 1998. Passages quoted in Coulomb’s original French are appended in an English translation. Professor Schofield is a Professor of Engineering, and a Fellow of Churchill College, in Cambridge; a Fellow of the Royal Society, the Royal Academy of Engineering, and the Institution of Civil Engineers (ICE), in London; he holds the US Army Outstanding Civilian Service Medal and the ICE James Alfred Ewing Gold Medal for Research.

Abstract; Coulomb’s Essay on limiting statics stated that newly remoulded sol has no cohesion. Critical state soil mechanics agrees with this principle, here called Coulomb’s law. The Mohr-Coulomb equation wrongly interpreted strength data. The two components of peak strength really are friction and interlocking.

1 Coulomb’s 1773 essay

1.1 Newly remoulded soil has no cohesion.

Heyman (1972) made an annotated translation of Coulomb’s Essay on statics, with comments on Coulomb’s references. His comments on the Mohr Coulomb equation, on limiting statics of soil, on the bending of beams, and on failure mechanisms of masonry arches with plastic hinges, are in the context of the upper and lower bound theorems of structural plasticity. He has historical notes on Coulomb’s work and time.

This paper is about a principle that Coulomb states three times in the Essay, that newly remoulded soil has no cohesion, which was not discussed at the time of Heyman’s translation, but which I now think so important that here I call it Coulomb’s Law.

The part of his Essay that is on earth pressure, is based on earlier work on friction and cohesion by Amontons, La Hire, and Musschenbroek. Not until Cauchy, see Timoshenko (1953), was it understood that the stress at a point in a continuum is a type of physical quantity that must be described by an array of numbers, stress components on planes at different angles through that point. For Coulomb, 50 years before Cauchy, friction and cohesion are properties available in material. Equilibrium is disturbed only if the shear force on a slip plane exceeds these two strength components. The cohesion component is proportional to a plane area but is independent of force normal to the plane. Friction is proportional to normal force on a plane but independent of its area.

Coulomb had learned that solid bodies offer the same resistance to separation of their parts in shear (cohesion) as in tension (adhesion). To test and confirm this teaching he made experiments on rock in tension and shear, and in his Essay wrote about his own experimental data as follows;

"qu'il falloit une plus grande force pour rompre le solide, lorsque cette force étoit dirigée suivant le plan de rupture, que lorsqu'elle étoit perpendiculaire à ce plan. Cependant, comme cette différence n'est ici que 1/44 du poids total, & qu'elle s'est trouvée souvent plus petite, je l'ai négligée dans la théorie que suit."

Coulomb's acceptance of this teaching led to his principle that there can be no cohesion in newly remoulded soil. In two Examples in sections XI and XIV, and half way through section XV, he states this principle, without emphasis, as follows;

"si l'on suppose l'adhérence nulle; ce qui a lieu dans les terres nouvellement remuées:"

"Si l'adhérence δ est supposée nulle, comme dans les terres nouvellement remuées:"

"Supposons, pour simplifier, $\delta = 0$, ce qui a lieu pour les terres nouvellement remuées;"

If adhesion is the same as cohesion then adhesive or cohesive strength are both destroyed by remoulding. Since in practice, ground was broken up with picks and broken soil was tipped from barrows as fill behind retaining walls, Coulomb would not count on any adhesion or cohesion in calculating the earth pressure of such fill. He has a clear principle on soil strength; if soil is newly remoulded then it has neither adhesion nor cohesion. He uses the word "suppose" as follows. Ground may have strength when intact and loose it at some unknown past time. If the ground is newly remoulded then in principle his design will not rely on soil adhesion or cohesion. As an engineer he must suppose that the "worst case" damage has taken place in the recent past.

Nowhere in the Essay does Coulomb write what is called "Coulomb's equation" by Terzaghi (1942). That equation probably had been taught to Coulomb in his engineering school. It is the earlier work of others; people like him “doivent seulement chercher à perfectionner ce qui des mains plus habiles ont crée.”

The lateral earth pressure equation written in section X, that geotechnical engineers should learn as Coulomb’s equation, is $A = ma^2 - \delta la$; the context is

“chercher la plus grand pression A pour l’empêcher de glisser … pour le triangle de la plus grande pression, par les règles de maximis minimis … l’on aura $A = ma^2 - \delta la$, m & l étant des coëfficiens constans, où il n’entre que des puissances de n; cette force A sera suffisante pour soutenir une masse.”

In Coulomb’s equation, a is the height of the wall. 1/n is the coefficient of friction. The coefficients l and m involve cohesion and friction. Coulomb notes that when friction and cohesion become zero, his equation gives the fluid pressure. This paper is concerned with a principle held by Coulomb on strength of soil, here stated as his law; that newly remoulded soil has no cohesion. This brief paper will show that Rankine’s work is consistent with this law; that Mohr’s explanation of peak strength is not credible and Terzaghi’s interprets clay peak strength data wrongly; that Taylor’s interlocking is right and critical state soil mechanics conforms to Coulomb’s law. A brief paper such as this can only advise engineers of a risk of an error of principle, and indicate a way forward for the future.

1.2 Rankine’s teaching on adhesion

Rankine worked with knowledge of the ellipse of stress. His Manual for civil engineers states that the strength with which soil resists shearing force

“arises partly from the friction between the grains, and partly from their mutual adhesion; which latter force is considerable in some kinds of earth, such as clay, especially when moist. But this adhesion is gradually destroyed by the action of air and moisture, and of the changes of the weather, and especially by alternate frost and thaw; so that its friction is the only force which can be relied on to produce permanent stability.”

“The permanent stability of earth, which is due to friction alone, is sufficient to maintain … an uniform slope, whose inclination to the horizon is the angle of repose, or angle whose tangent is the co-efficient of friction.”

“The properties of earth with respect to adhesion and friction are so variable that (we obtain) data either by observation of existing earthworks in the same stratum or by experiment.”

Rankine’s evaluation of terrain was simple. Ground that stood for a long time with a vertical face was soft rock; ground that in the long term stood with an angle of repose was soil. Rankine hoped to find a solution for the problem of stress in slopes of general profile by finding conjugate surfaces analogous to isothermals in the problem of steady heat flow. He was criticised by Boussinesq (1874) for an error by which he derived the heat equation rather than the wave equation. It does not affect equations for which Rankine is well known. His achievement was to combine Lamé's ellipse of stress with limiting friction to find limits for the stress in a granular continuum. In this he is very close to Coulomb, who writes in section X

"Ainsi, il résulte que la différence entre la pression des fluides dont le frottement & la cohésion sont nuls, & de ceux où ces quantités ne doivent point être négligées, consiste en ce que dans les premiers, le côte .. du vase qui les contient ne peut être soutenu que par une seule force, au lieu que dans les autres, il y a une infinité de forces contenues entre les limites A & A′, qui ne troubleront point l'état de repos."

Rankine notes that adhesion is useful in helping ground stand up in temporary work but plays no part in his design, which is consistent with Coulomb's law. Adhesive or cemented strength may arise as small creep strains occur and as bonds develop in undisturbed sedimentary soils, but any such bonds are, in Rankine's view, destroyed by weathering, and in Coulomb's view destroyed by remoulding. These two founders of soil mechanics are in agreement.

2 Mohr's and Terzaghi's teaching.

2.1 Stress circle and limiting envelope

Culmann, (see Timoshenko) represented states of plane stress by circles. Mohr proposed that the envelopes to stress circles define material strengths. Voigt and Karman found this untrue for crystals or sandstone. Mohr's strength theory was followed by fracture mechanics, but Terzaghi had taught Mohr's theory, and geotechnical engineering has retained it through the 20th century. At the start of the 21st century there is a need to review theories of soil strength and methods of validation of design

A problem for teachers is that soil mechanics is the only field of engineering where students must think that material behaviour depends on vectors of stress on particular planes. Mohr's plane stress circle offers two planes with identical stress conditions. Students see a fault plane in a specimen that has failed, but why did the fault chose one of the planes, and is the measured inclination of the fault plane precisely at the predicted angle? It has not been satisfactory to teach Mohr's theory when there are questions it does not answer. It is not credible that peak strength can occur at the same moment of time everywhere on an extensive surface through a body of soil under limiting stress. In all soil tests we see that strain affects strength. A theory that takes no account of strain can be true of critical state plastic flow with large strains but it can not be credible teaching for soil failure at peak strength.

2.2 Limiting statics of earth

Because in calculations of limiting statics it makes no difference whether a wall rotates about a hinge point at the top or the bottom, Terzaghi called such calculation a “fallacy”; his experience showed that earth pressure on trench supports depends on strains.

The Mohr Coulomb equation and the two equations of plane equilibrium are a system of three equations with three unknowns that can in principle be solved by the method of characteristics for given boundary conditions. Where in the solution can strains be introduced? The two equations of equilibrium are fixed. The Mohr Coulomb equation is the only place where strains can come in. The limiting statics calculations are correct if “friction” and “cohesion” of soil are properties that do not depend on strains, which applies only to flow at critical states.

2.3 Terzaghi’s “cohesion”.

Terzaghi wrote out his teaching as follows;

“the data for making a stability calculation pertaining to clays can at present be obtained only by means of the following purely empirical procedure. We test the clay in the laboratory under conditions of pressure and drainage similar to those under which the shear failure is likely to occur in the field”

“If we dig into a bed of dry or of completely immersed sand, the material at the sides of the excavation slides towards the bottom. This behaviour indicates the complete absence of a bond between the individual sand particles. The sliding material does not come to rest until the angle of inclination of the slopes becomes equal to a certain angle known as the angle of repose. The angle of repose of dry sand as well as that of completely immersed sand is independent of the height of a slope. On the other hand a trench 20 to 30 feet deep with unsupported vertical sides can be excavated in stiff plastic clay. This fact indicates the existence of a firm bond between the clay particles. However, as soon as the depth of the trench exceeds a certain value, dependant upon the intensity of the bond between clay particles, the sides of the trench fail and the slope of the debris which covers the bottom of the cut after failure is far from vertical. The bond between the soil particles is called cohesion. No definite angle of repose can be assigned to a soil with cohesion, because the steepest slope at which such a soil can stand decreases with increasing height of the slope. Even sand, if it is moist, has some cohesion.”

Terzaghi appeals to field experience to explain stiff clay behaviour and therefore links cohesion with the state of an aged clay in which bonds have formed after a prolonged period of creep. He writes about total stress, not effective stress. If he had made a vertically sided cylinder of newly remoulded clay by compaction or consolidation, 10 or 15 cm high, he could have left it standing on his laboratory bench with an air water interface on all faces of the cylinder. With suction in the pores it would not have failed under self weight because frictional strength would have been mobilised by effective stress. But tensile strength of pore water is not part of the strength of effectively stressed soil. If he had filled a sink in his laboratory with water, picked up that cylinder of newly remoulded clay and immersed it in the sink, he could have left it under water for his students to observe. Before long, the suction in the pores would have drawn water into the faces of the cylinder of soil. Soil particles would begin to fall off the vertical faces, demonstrating Coulomb's law to students who saw a heap of soil under water with fully softened clay soil slopes at an angle of repose. Rankine wrote about these matters as follows

"One of the effects of the temporary stability due to adhesion is seen in the figure of the surface left after a "slip" has taken place in earthwork. That surface is not an uniform slope, inclined at the angle of repose, but is concave in its vertical section, being vertical at its upper edge, and becoming less and less steep downwards. It is not capable, however of preserving that figure; for the action of the weather, by gradually destroying the adhesion of the earth, causes the steep upper part of the concave face to crumble down, so that the whole tends to assume an uniform slope in the end." In that passage Rankine accepts both that chemical weathering can destroy the adhesive bonds in aged clay and that the fallen soil has no "true" cohesion.

2.1 True cohesion and friction

Tests of remoulded clay in Terzaghi's shear box in Vienna gave drained peak strength data which were interpreted by Terzaghi as showing "true" cohesion. A feature of those tests not noted by Terzaghi is that the Mohr Coulomb equation fits data in certain over consolidation states only, (the solid line in Figure 1.)

See attached Figure 1.

The development of faults or surfaces of rupture through a soil body was observed by Coulomb and described by the Mohr Coulomb equation. It is one of three possible regimes of behaviour, a bifurcation that occurs only in soil in particular states, much as turbulent flow is a regime in which there is a particular instability in a fluid. Three regimes of behaviour are observed in aggregates of frictional soil particles are as follows:

(i) when soil is lightly over consolidated, to the right of the solid line, it can flow as a ductile plastic continuum;
(ii) between the over consolidation ratios indicated by the solid line soil exhibits faults, with a layer of gouge material on the slip surface softening progressively from peak to critical state strength; these are

the slip planes that Coulomb observed and that Mohr associated with limiting stress envelopes;

(iii) very highly over consolidated soil in states to the left of the solid line exhibits tensile fissures; if there is a high hydraulic gradient across a soil body in these states, that was initially stiff but becomes fissured, it is rapidly transformed into a clastic debris flow as it liquefies.

How can soil exhibit apparent cohesion as it flows, exhibit apparent cohesion at peak strength, and satisfy Coulomb's law? The interlocking in an aggregate of solid particles explains this behaviour.

3 Taylor's interlocking

3.1 Taylor (1948) and his sand tests at MIT

In drained sand shear box tests, loose sand ultimate shear strength was $\tau = \mu \sigma'$, where τ and σ' are stress components on the shear plane and μ is the coefficient of friction. Dense sand dilated when sheared. Taylor called this effect interlocking. and calculated the work due to friction and interlocking as two components of peak strength as follows;

$$(\tau\, dx) = (\mu\, \sigma'\, dx) + (\sigma'\, dy) \ .$$

where dx and dy are displacement components. The ratio of (peak strength) / (effective normal stress) is

$$(\tau / \sigma') = (\mu) + (dy / dx) \ ;$$

(strength ratio) is (friction) plus (interlocking). After peak strength, dense sand on fault planes dilates and strength falls back to critical state values.

Interlocking applies to all soil; to clay and to gravel. The fact that soil on the dry side of the critical states dilates and sucks in water after peak strength means that interlocking provides the component of peak strength that give rise to apparent cohesion.

On the dry side of critical states we cannot get test data of the clay water content in the region of failure at the moment of failure, or of interlocking there. However, Roscoe, Schofield, and Wroth displayed Hvorslev's equation for peak strength of remoulded clay as a three dimensional surface in (τ,σ',e) space, and drained shear box tests as paths crossing the surface up to a critical state line, an edge of the Hvorslev surface. As the water content increased, strength fell from peak values. Paths on the wet side of critical states also move towards the critical state line, but that soil on the wet side hardens and so specimens are stable and give good research data.

3.2 Critical states and apparent cohesion

Undrained apparent cohesion of soil in a critical state is not due to adhesion between soil particles, but is a product of effective stress times critical state friction. Why is soil ductile like metal? When metal deforms any atom can exchange electrons and form bonds with any nearby atom. It is different in soil at a critical state. Soil particles change partners but they do not form bonds while flowing. In undrained shearing at a critical state the effective stress is constant, and the soil strength remains constant because the friction angle of soil at a critical state has a constant critical state value during flow. In remoulded soft soil the small clay mineral particles and chemicals cause pore water suction. With zero total stress, suction in pores equals positive effective stress among particles, giving remoulded soft clay its (apparent cohesion) = (suction) times (friction).

Terzaghi's analysis of test data considered Mohr's circles at the end of tests. Different test specimens had slightly different densities and he accepted that some scatter was inevitable in the different circles to which envelopes were fitted. Where Terzaghi's soil mechanics is inaccurate, the study of test paths created the possibility of greater accuracy. We had a series of data points relating to a single specimen. What sort of shape could we expect a path to have?

A typical early Imperial College drained test on the wet side of critical states had a duration that ensured that the pore pressure gradient from the middle to the ends was low at failure. The early stages of such tests took place relatively quickly and drainage was poor. The paths published in 1958 looked parabolic,

3.2 Elastic energy and work in shear

This shape did not prove to be reliable. Thurairajah made triaxial tests of kaolin clay and calculated what work went into Taylor's dilation and what went into change of elastic energy in the effectively stressed soil. The result of Thurairajah's work with both drained and undrained triaxial tests was surprising. He found that the work dissipated in plastic deformation of kaolin equals the product of the effective mean normal stress p' times M, the coefficient of friction at critical states, and that this result applied not only when paths reached critical states but at all stages of the test paths. As a test approaches critical states the soil particle packing can change, and the stored elastic energy can change. In an increment of deformation $\delta\varepsilon$; the work done is only $M\,p'\,\delta\varepsilon$ and Taylor's calculation of work dissipated at peak strength of sand applies to all stages of yielding of soil on all test paths. There is no need for separate consideration of change of volume or elastic stored energy in a plastic strain increment $\delta\varepsilon$ because they depend on the stress in that increment, as the associated flow rule of theory of plasticity predicts.

3.3 An alternative interpretation of peak strengths

This gave an alternative to Terzaghi's interpretation. The two components of peak strength are friction and interlocking. The explanation of the line of peak strengths was that on the dry side of critical states, (to the left of C in Figure 2), soil dilates with increasing strength, (see Taylor's Figures 14.2 and 14.10 and read his sections 14.9 and 14.10).

See attached Figure 2.

3.4 The yielding of wet clay

The associated flow rule gave the yield locus slope. By integration, Roscoe and Schofield got the yield locus for an ideal soil in states wetter than critical. The theoretical shape for an undrained test path on the wet side of critical states fitted creep test data.

A strong feature of this original yield locus is a sharp point on the space diagonal in effective stress space. It was a consequence of Taylor's dissipation function. If the only way that work is dissipated is in shear distortion then there is a point on the yield locus at which spherical stress causes shear distortion, and a wide range of shear distortions for which this ideal soil exhibits fluid pressures. I used the name cam-clay to describe this ideal soil, Schofield and Wroth (1968). Although Roscoe and Burland then modified the dissipation function to make the yield surface elliptical and to give a more typical lateral pressure at rest of about 0.7, the original cam clay model had followed from Taylor's fundamental work and was supported by data. Rather than introduce successive changes in models, each of which has particular use, I wanted simple statements. Terzaghi had made an error; teaching should be based on right principles, and design needed validation, in future.

3.5 Design strength

A model of the cam clay type explains data of test paths on the wet side of critical states with only friction and plastic volume contraction, and without any adhesion or cohesion. Friction is the only force that can be relied on to produce permanent stability. Interlocking frictional particles in solid to solid contact, with plastic volume changes during shear distortion, will have peak strengths that fit angles of friction higher than the critical state angle. It is a geometrical matter, much as in the case of the arch, as Heyman notes in his discussion of masonry structures. Arching and interlocking are evident in earth pressure tests, but it is not clear what strains and displacements will mobilise peak strength.

Critical State Soil Mechanics taught reliance only on large strain critical state angles of friction in design and quoted Coulomb's factor of safety of 1.25 as a value consistent with Coulomb's principles, set out in Section X, “d’un quart en sus de celle qu’exige l’équilibre”

“M. le maréchal de Vauban, dans presque toutes les places qu’il a fait construire, a donné 5 pieds de largeur au cordon, sur 1/5 de talud. Comme les revêtemens construits par cette homme célèbre, passent rarement 40 pieds, sa pratique se trouve dans ce cas assez d’accord avec notre dernière formule. Il est vrai cependant que M. de Vauban ajoute des contre-forts à ses murs; mais cette augmentation de solidité ne doit point être regardée comme superflue dans les fortifications, dont les enveloppes ne doivent point être culbultées par le premier coup de canon.”

However, although it is safe to teach students design based on critical state strength, many experienced engineers have based design on Terzaghi’s teaching and on field test data, and they could not reject the Mohr Coulomb equation until they had a way to proceed in practice. Experiments show that lateral earth pressure development fits a limiting stability calculation with an apparent angle of friction greater than the critical state angle, but tests are needed to find the strain required in soil structure interaction.

Centrifuge modelling offered an alternative to full scale tests if we could develop methods of measurement of pore water pressures, and match techniques adopted at full scale. Experiments in geotechnical centrifuges test newly remoulded soil in the laboratory under conditions of pressure and drainage similar to those under which the shear failure is likely to occur in the field, and in that respect they satisfy Terzaghi’s teaching. It has taken time to develop reliable techniques and affordable centrifuges, but their time has now come.

Design based on small strain properties of intact soil at working stress ratios higher than the critical state can be safe. Calculations using friction and interlocking to check safety and serviceability at working load, with partial drainage, can be validated by centrifuge model tests. It will be seen in contributions (to the meeting in Paris on 19 MAY 1998) that follow this paper there is the same need for serious experimental mechanics now as there was when Coulomb began his career as an army engineer and an experimental scientist.

4 CONCLUSION

Coulomb was elected a membre de la section de physique expérimental of the Institute in 1795, by then a notable physicist. At the time that he wrote his Essay twenty years earlier he wrote only as an engineer after thirteen years of service in the war in America, with modesty and with a hope

“de rendre les principes dont je me suis servi assez clairs pour qu’un Artiste un peu instruit pût les entendre & s’en servier”

“Ce Mémoire, composé depuis quelques années, n’étoit d’abord destiné qu’à mon usage particulier, dans les différens travaux dont je suis chargé par mon état; si j’ose le présenter à cette Académie, c’est qu’elle accueille toujours avec bonté le plus foible essai, lorsqu’il a l’utilité pour objet.”

In adopting the Mohr Coulomb equation Terzaghi made the error of regarding apparent cohesion as a soil property independent of strains. More utility can come from Coulomb’s work, if we teach principles that Coulomb used, and if we assess soil strength with due regard to Coulomb’s law. The geotechnical centrifuge is a good apparatus in which to test the interaction of a structure with newly remoulded soil.

5 REFERENCES

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6 APPENDIX

Heyman’s translations of passages quoted above.

that a larger force was needed to break the body when the force was directed along the fracture plane than when it was perpendicular to that plane. However, since the difference is here only 1/44 of the total weight, and was often found to be smaller, I have neglected it in the following theory.

(Assuming) … the cohesion is zero, as for newly turned soils,

If the cohesion δ is assumed to be zero, as for newly turned soils,

Assume for simplicity that $\delta = 0$, as for newly turned soils;

should try only to only to perfect what more capable hands have created.

That triangle must be sought , which requires the largest force A to prevent slip ..the triangle of greatest thrust, … from the rules of maximum and minimum….it will be found that $A = ma^2 - \delta la$, where m and l are constant coefficients containing only powers of n; this force will be large enough to sustain (a).. mass

Thus it follows that the difference between forces in fluids for which friction and cohesion are zero and those for which these quantities can not be neglected is that for the former the side .. of the vessel containing them can be supported only by a unique force, while for the latter there is an infinite number of forces lying between the limits of A and A' which will not disturb equilibrium.

a quarter above that required for equilibrium

Marshal Vauban, in almost all the fortresses he built, made the ridge 5 ft. wide, with a batter of 1/5. Since the retaining walls built by this famous man were rarely higher than 40 ft., his practice is in this case in reasonable agreement with our last formula. It is true however that Vauban added butresses to his walls; but this strengthening should not be thought superflous in fortifications, of which the enceintes ought not to fall to the first canon ball.

to make the principles I have used sufficiently clear that a workman with a little learning could understand and use them.

This paper, written several years ago, was originally meant only for my own use, in the different tasks in which I was engaged in my profession; if I dare to present it to this Academy, it is because the weakest work is always received kindly by it if the subject is of practical use.

7 FIGURES

"Mohr Coloumb" error figures

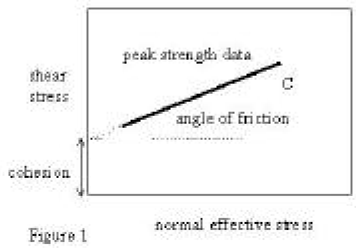

Figure 1

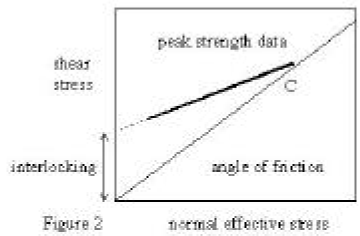

Figure 2