Original Cam-clay

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ORIGINAL CAM-CLAY

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ABSTRACT: The peak strengths observed in shear box tests of silty clay can be interpreted either as evidence that clay mineral particles contribute a cohesive bond, or that silt particles contribute interlocking, in addition to the frictional strength due to effective mean normal stress between the soil particles. Soft ground appears to behave as a ductile plastic continuum. Assuming that it satisfies the associated flow rule of the theory of plasticity, and that plastic volume change and friction between particles are both involved in the strength of soft ground, it is possible to derive a theoretical yield locus for an ideal soil called Cam-clay. This ideal soil exhibits behaviour often attributed to clay, even though it is simply an aggregate of interlocking frictional particles.

1. TAYLOR'S INTERLOCKING

In his analysis of shear box tests of dry sands D.W. Taylor[1] proposed that two factors made contributions to strength, of soil. One factor was the frictional resistance between particles as they slipped during shear distortion of soil. A second factor he called interlocking; this factor required work to be done to cause volume increase during shear distortion. As an example, consider a test on dense sand Figure 1(a). At peak strength, while the relative displacement of the two halves of the shear box increases by an amount dx , Figure 1(b), there is an increase in sample thickness and a separation of the two halves of the box by an amount dy . The quantity dy/dx indicates the rate of dilation of sand in the shear zone. It is a measure of the interlocking. When dy/dx reaches a maximum the strength reaches a peak. As dx increases and dy/dx tends to zero, the two halves of the dense sand body in the shear box eventually slip relative to each other at a constant stress $\mu\,\sigma$. A very thin layer of loosened sand on the slip plane has then reached a critical state, and there is no more interlocking.

We can calculate the rate at which work is done at peak strength as

$$\tau\,dx \quad = \quad \mu\,\sigma\,dx \quad + \quad \sigma\,dy\,; \qquad (1)$$

work in friction and dilation .

We can divide through by σ dx and obtain an equation

$$\tau/\sigma \quad = \quad \mu \quad + \quad dy/dx\,; \qquad (2)$$

strength = friction. + interlocking .

The coefficient of friction μ in Taylor's example was $\mu = 0.475$; the calculation of interlocking at peak strength was $dy/dx = 0.17$; the peak strength was $\tau/\sigma = 0.645$. These results would not have changed if the entire shear box-had been immersed in water and it had been a drained shear box test on saturated sand.

2. INTERPRETATION OF DATA OF PEAK STRENGTH OF SOIL

Drained shear box tests can be performed on water-saturated, immersed specimens of any soil. A number of tests can be done on specimens of the same soil, each at a different normal effective pressure σ'. The data of peak drained strengths can be plotted as a line, Figure 2(a)

$$\tau = c + \sigma' \tan\phi\,. \qquad (3)$$

It is possible to interpret this line as being evidence that the soil particles adhere to each other and that c is cohesion. Alternatively, the same set of data can be seen as evidence that the soil has a constant coefficient of friction, but that there is a variable amount of interlocking, Figure 2(b).

At any given specific volume v (the volume of space occupied by a unit volume of solid soil particles) there is a critical effective pressure σ'_c at which the soil will deform in shear without dilation, and so

$$c + \sigma'_c \tan\phi = \mu\, \sigma'_c ,$$

hence

$$c = (\mu - \tan\phi)\, \sigma'_c . \tag{4}$$

For specimens tested under an effective pressure $\sigma' < \sigma'_c$, the dilatancy could be expressed as

$$\frac{dy}{dx} = \frac{c}{\sigma'} - (\mu - \tan\phi)$$

$$= (\mu - \tan\phi)\left(\frac{\sigma'_c}{\sigma'} - 1\right). \tag{5}$$

The dilatancy then satisfies the alternative explanation of peak strengths as being due to interlocking and not due to cohesion. There are difficulties both with this equation for dilatancy, and with the test data, as effective pressure σ' tends towards zero . At low effective pressure, data of direct shear box tests are unreliable. A body of newly remoulded silty clay is held together by the surface tension that creates a positive effective stress between silt particles. This positive effective stress is equal to the negative suction in the pore water within the clay body. If the body is immersed in water the surface tension and the suction eventually will no longer hold the particles together. Bodies of water-saturated, reconstituted, clay soil which are immersed in water become very soft.

A study of the strength of overconsolidated clay specimens in direct shear box tests, in Terzaghi's laboratory in Vienna, was undertaken by Hvorslev[2]. He was able to observe and control the water content in the shear zone and the stress components on the slip planes before failure, but he was not able to observe dilatancy and change of water content in the very thin failure plane after failure. Terzaghi interpreted Hvorslev's data as evidence of "true cohesion" which bonded clay particles together. He did not consider the possibility of an interpretation involving interlocking.

In an attempt to advance testing techniques Roscoe, in Cambridge, England, developed a simple shear apparatus in which his successive students attempted to study the changes in conditions in the shear zone both in sand and in clay soils. In 1958 a study of the yielding of soil based on some Cambridge data of the simple shear <apparatus tests, and on much more extensive data of triaxial tests at Imperial College, London, led to publication of the critical state concept (Roscoe, Schofield, and Wroth[3]). If v is the specific volume under an effective normal pressure σ' , then for any one particular soil the critical states satisfy an equation

$$v + \lambda \ln \sigma' = \text{constant}$$

where (v, σ') is a combination of values of specific volume and effective pressure that define one of the critical states and λ is a soil parameter. Roscoe hoped that it would be possible to observe changes of specific volume during uniform shear distortion after failure at peak strength in his simple shear apparatus, as the critical states were approached at large strain. This did not prove possible. The triaxial apparatus proved to be better for study of the yielding of soil, and in the following sections of this paper the discussion will be confined to data of triaxial tests on cylindrical samples of soil.

3. ELASTIC STATES AND CRITICAL STATES OF TRIAXIAL TEST SPECIMENS

A cylinder of water-saturated sand in a triaxial cell, Figure 3, carries total axial stress σ_a , total radial stress σ_r , and has a pore water pressure u . Since the effective stresses are $\sigma' = (\sigma - u)$ we can define the mean normal effective. stress as

$$p' = \tfrac{1}{3}(\sigma_a + 2\sigma_r) - u = (\sigma'_a + 2\sigma'_r)/3 ,$$

and the triaxial deviator stress as

$$q = (\sigma_a - \sigma_r) = (\sigma'_a - \sigma'_r) . \qquad (6)$$

For a given water-saturated aggregate of interlocking soil particles with $q = 0$ it is found that as p' slowly varies there is a non-linear "elastic" compression and swelling of the aggregate during which data will approximately fit an equation

$$v + \kappa \ln p' = \text{constant} = v_\kappa . \qquad (7)$$

Here κ is a soil parameter, and v_κ represents one particular aggregate of particles during drained compression and swelling without relative slip of soil particles. When there is plastic irrecoverable slip within the aggregate of particles then there is a consequent change of v_κ .

Even when a triaxial test begins with a soil specimen in an elastic state, as q increases eventually there is plastic yielding of the soil as the test progresses and the strains increase. Data of triaxial tests indicate that at large strains, saturated soil specimens come into "critical states" defined by $p'_c = p'(v)$, $q_c = q(v)$, where

$$v + \lambda \ln p'_c = \Gamma ,$$

$$q_c = M p'_c . \qquad (8)$$

If we define a parameter $v_\lambda = v + \lambda \ln p'$ then there is a region in which $v_\lambda < \Gamma$, called the "dry" side of critical state; this is the region in which soil fails with peak strength on a slip surface as described above. The region in which $v_\lambda > \Gamma$ is called the "wet" side of critical states, and is the region in which soil yields as a ductile plastic continuum. The "Cam-clay" equation which is the subject of this paper applies to soil in this latter region.

4. THE ASSOCIATED FLOW RULE OF THE THEORY OF PLASTICITY

In general, engineering students are taught to derive a plastic yield locus by considering limiting equilibrium, and are not taught to consider the dissipation of work and the plastic flow that is associated with plastic yielding. To explain the concept of "associated plastic flow" a simple example will be given. The problem considered is the plane bearing capacity of a plate which supports a combination of a vertical load, V , and a moment, M . For the sake of simplicity, the plastic body on which the plate rests will be a level surface covered with many plastic compression elements. They remain rigid under any pressure less than f , and they compress vertically with plastic slip when the pressure reaches and is held at f .

Students are taught to derive a yield function and a locus from consideration of limiting statical equilibrium in the following manner. In Figure 4 the plate of width 2b rotates clockwise through an angle $d\theta$, about a centre of rotation that is a distance 2a from the left-hand edge of the plate. The foundation reaction is a uniform limiting pressure f below the portion of the plate that is indented into the surface. From equilibrium, for clockwise rotation

$$V = (2b - 2a) f , \quad \text{and} \quad M = (b - a) 2af ,$$

and for anti clockwise rotation

$$V = 2af , \qquad \text{and} \quad M = -(2b - 2a)\, af . \qquad (9)$$

Introducing $V_o = 2bf$ and eliminating a from these equation gives a yield function

$$F = \pm\left(\frac{M}{bV_o}\right) + \left(\frac{V}{V_o}\right) - \left(\frac{V^2}{V_o^2}\right) = 0 \qquad (10)$$

hence a yield locus can be drawn which consists of two segments of parabolic curves, Figure 5.

Just as in Figure 1 there were two components of displacement (dx, dy, relative displacement of the two halves of the shear box) so in Figure 4 there are two components of displacement ds, $d\theta$, of the plate. The settlement ds of the vertical load V , and the rotation of $d\theta$ of the moment M , are related by an equation

$$ds = (b - 2a)\, d\theta\,. \qquad (11)$$

The work done in yielding can be calculated as the vector product of the loads times the displacements. This should equal the work dissipated in the plastic indentation of the foundation, which can be calculated to be the product of the limiting pressure f times the swept volume of the indentation. This calculation can be shown to be correct, for example for a clockwise rotation, since

$$\left.\begin{aligned} V ds + M d\theta &= (2b - 2a.)\, f\, [(b - 2a) + a]\, d\theta \\ &= \left[(2b - 2a)\, d\theta\,.\,(2b - 2a)/2\right] f = [\text{swept volume}]\,.\,f\,. \end{aligned}\right\} \qquad (12)$$

This check calculation confirms that the load and displacement parameters are "work conjugate",

In Figure 5 the axes are marked with dimensionless parameters (V/V,) and (M/bV_o) , and also with displacement parameters (ds/b) $(d\theta)$. The figure demonstrates the "associated flow rule" of the theory of plasticity It shows the vectors of plastic flow increments to be orthogonal to the yield locus. Four examples of associated flow vectors are shown in Figure 5 and in each case there is a link between the combined load vector and the flow vector, as follows:

(A) a clockwise rotation $d\theta$ about the centre of the plate, with zero vertical settlement $ds = 0$, is induced by a combination of load $V = bf = \frac{1}{2}V_o$, and moment $M = + fb^2/2$;

(B) a clockwise rotation $d\theta$ about the left edge of the plate, with vertical settlement $ds = + bd\theta$, is one possible displacement mechanism that can occur under the maximum vertical load $V = 2bf = V_o$; there is uncertainty about the rotation because

(C) an anti-clockwise rotation $- d\theta$ about the right edge of the plate, also with vertical settlement $ds = + bd\theta$, is another possible displacement mechanism, and any outward directed vector between B and C represents a possible displacement mechanism, under $V = V_o$;

(D) an anti-clockwise rotation $- d\theta$ about a point at a distance $3b/2$ from the right edge of the plate with a vertical rise, (a negative settlement) $ds = - bd\theta/2$, is induced by a combination of vertical load $V = fb/2$, and moment $M = - 3fb^2/8$.

In each case the yield locus not only defines the combinations of loads that will cause yielding. The yield locus also defines the plastic flow that will be associated with a particular combination of loads. There is uncertainty about the rotation when $V = V_o$.

For any flow that is free from vorticity there is a potential function such that the normals to that function define the flow vectors. For plastic yielding the yield locus serves as the potential function.

5. YIELDING OF SOIL ON THE "WET" SIDE OF CRITICAL STATES

Soil on the "wet" side of critical states is observed to yield as a ductile plastic continuum. We seek a potential function for plastic flow of soil in those states.

For the triaxial test specimen, Figure 3, the generalised load and load increment vectors are (q, p') and $(\delta q, \delta p')$. The resulting plastic deformations are a change of specific volume (evident in a back pressured burette), and an axial strain

$$\delta\varepsilon_a = \delta H/H\,. \tag{13}$$

During isotropic compression with $\delta\varepsilon_a = \delta\varepsilon_r$, the volumetric strain is

$$\delta v/v = \delta\varepsilon_v = (\delta\varepsilon_a + 2\delta\varepsilon_r) = 3\delta\varepsilon_a \tag{14}$$

so that $\delta\varepsilon_a = \frac{1}{3}\,\delta\varepsilon_v$. During general deformation we can separate the volumetric strain from the distortional strain if we subtract $\frac{1}{3}\,\delta\varepsilon_v$ from $\delta\varepsilon_a$. We therefore define triaxial shear strain as

$$\delta\varepsilon_s = \delta\varepsilon_a - \tfrac{1}{3}\,\delta\varepsilon_v = \tfrac{2}{3}\,(\delta\varepsilon_a - \delta\varepsilon_r)\,. \tag{15}$$

We can confirm that the plastic strain increment vector $(\delta\varepsilon_s, \delta\varepsilon_v)$ is work conjugate with the load vector (q, p') by calculating

$$\left.\begin{aligned} \delta W &= p'\,\delta\varepsilon_v + q\,\delta\varepsilon_s \\ &= \tfrac{1}{3}\,(\sigma_a' + 2\sigma_r')\,(\delta\varepsilon_a + 2\delta\varepsilon_r) + (\sigma_a' - \sigma_r')\,\tfrac{2}{3}\,(\delta\varepsilon_a - \delta\varepsilon_r) \\ &= \sigma_a'\,\delta\varepsilon_a + \sigma_r'\,\delta\varepsilon_r\,. \end{aligned}\right\} \tag{16}$$

• In 1962 I derived a theoretical yield locus for ideal soil "wetter than critical" as follows. It appeared likely, and I assumed it to be the case, that the dissipation of work during a general deformation of soil in states wetter than critical was

$$p'\,\delta\varepsilon_v + q\,\delta\varepsilon_s = \delta W = Mp'\,\delta\varepsilon_s\,, \tag{17}$$

where M was a generalised coefficient of friction (capital μ). This dissipation function can be regarded simply as generalisation of equation (1). It should be noted that both Taylor's equation (1) and my dissipation function equation (17) assume that when there is some combination of volume change (dy or $\delta\varepsilon_v$) and of shear distortion (dx or $\delta\varepsilon_s$) it is the shear strain thatdetermines the dissipation rate. The dilation or volume change is a geometrical consequence of interlocking, and does not need to appear explicitly in the dissipation function. Subsequently Burland introduced modified assumptions but in this paper I will not consider modified Cam-clay.

The original idea was very simple. The yield locus must be such that each associated plastic flow vector $(\delta\varepsilon_v, \delta\varepsilon_s)$ would be orthogonal to the tangent to the yield locus, hence locally

$$\delta\varepsilon_v/\delta\varepsilon_s = -\,dq/dp'\,. \tag{18}$$

In the manner followed in equation (2), equation (17) can be divided by $p'\delta\varepsilon_s$ to obtain an equation

$$\frac{\delta\varepsilon_v}{\delta\varepsilon_s} + \frac{q}{p'} = M\,, \tag{19}$$

into which equation (18) can be substituted to give

$$\frac{q}{p'} - \frac{dq}{dp'} = M\,. \tag{20}$$

This equation is integrable. Introducing the stress ratio q/p' = η , and

$$\frac{d\eta}{dp'} = \frac{1}{p'}\left(\frac{dq}{dp'} - \frac{q}{p'}\right) = -\frac{M}{p'}$$

gives $d\eta/M + dp'/p' = 0$. The locus has a horizontal tangent at B ,

$$\text{hence} \quad \frac{\eta}{M} = \frac{q}{Mp'} = 1 - \ln\frac{p'}{p'_c} \tag{21}$$

This is the desired equation. From equation (21) we can draw the yield locus BF in Figure 6(i) for soil in "states wetter than critical". In 1963 when this equation was first published[4] I called the ideal soil "wet-clay". Later, in the lecture notes that I issued to final year Cambridge undergraduates in January 1966, I substituted the name "Cam-clay" for the original "wet-clay". Roscoe and Burland[5] used the name "modified Cam-clay" and in the subsequent literature[6] it has proved helpful to introduce the words "original Cam-clay" to distinguish between the original and the modified assumptions.

It follows from the original equation (21) that as the effective pressure p' increases so the stress ratio η at which the soil yields will fall. When the effective pressure reaches a value

$$p' = 2.73\, p'_c \tag{22}$$

there will be plastic volume change with $q = 0$. Although there is no shear stress there can be chaotic shear distortion. When the ideal soil is subjected to isotropic plastic compression and reaches point F in Figure 6(i) it is in a state somewhat similar to the end point of the yield locus shown in Figure 5, where the associated flow vector can be anywhere between B and C . Within a triaxial test specimen of Cam-clay there can be some zones that slip in one direction and other zones that slip in a different direction.

While there is uncertainty about the shear distortion when original Cam-clay reaches the point F , there is no uncertainty about v_κ or v_λ . In Figures 6(i) and (ii) the points B, C, D are on the critical state line, equation (8). The points A', B', G', F' are on a swelling and recompression line, equation (7). At F , $q = 0$ and $\ln(p'/p'_c) = 1$, so in Figure 6 (iii) the length $B''F'' = 1$. At F" the slope of the line $F''H''$ is $(\lambda - \kappa)$ so $H''B'' = (\lambda - \kappa)$. This spacing between the line of isotropic plastic compression and the critical state line is a prediction that follows directly from the assumptions on which the original Cam-clay equation is based.

6. UNDRAINED AND DRAINED TEST PATHS

In Figure 6(ii) the critical state line B'C'D' has equation

$$v_\lambda = v + \lambda \ln p' = v_\kappa + (\lambda - \kappa)\ln p' = \Gamma$$

and the isotropic plastic compression line H'F' has equation

$$v_\lambda = v_\kappa + (\lambda - \kappa)\ln p' = \Gamma + (\lambda - \kappa) .$$

Between these two lines a third line G'E' represents all states of Cam-clay for which η has a particular value shown by the slope of the line GE in Figure 6(i). The spacing between B"C" and $G''E''$ can be seen in Figure 6(iii) and in equation (21). For any fixed value of η

$$\ln(p'/p'_c) = B''G'' = (1 - \eta/M) , \tag{23}$$

$$\text{hence} \quad B''J'' = (\lambda - \kappa)(1 - \eta/M) = E'D' . \tag{24}$$

So the line $J''G''E''$ in Figure 6(iii) has equation

$$v_\lambda = \Gamma + (\lambda - \kappa)(1 - \eta/M) . \tag{25}$$

In Figures 6(i), (ii) and (iii), two alternative test paths for triaxial test specimens are shown. The lines FD, F'D', $F''D''$ are for a test with p' = constant , which was a type of test first undertaken by Murayama's group in Kyoto, in which Shibata[7] found that as η increased there was a linear decrease of specific volume. The lines FC, F'C', F"C" are for a test with v = constant . This is a more common test – the undrained triaxial test with pore pressure measurement.

In Figure 7 all test paths are reduced to a single line in a plot of v_λ against η . People who perform triaxial tests will find that this is not a difficult way to plot their test data. The value of λ is easily and accurately obtained from data of soil classification tests and values of (p' v) are easily transformed into values of $v_\lambda = v + \lambda \ln p'$. The values of (q, p') give $\eta = q/p'$. This is a method of plotting test data that Schofield and Wroths proposed in 1968. It applies to the region with $\Gamma \leq v_\lambda \leq \Gamma + \lambda - \kappa$ labelled I in Figure 6. The extension of this plot into the region of Coulomb rupture labelled II and the region of tensile fracture labelled III, the regions with $v_\lambda \leq \Gamma$, was discussed by Schofield[9] but is not relevant here.

7. CONCLUSION

The Cam-clay equation is derived from the simple assumptions that any aggregate of interlocking soil particles will dissipate work in Taylor's manner, and that soil in states wetter than critical will be a ductile plastic continuum satisfying the associated flow rule. The Cam-clay equation explains the behaviour of soft ground without any need for assumptions about chemical bonds between clay mineral particles. It is a mechanical model for the behaviour of water-saturated silty soil. It is a simple model, and it does not address the important problems of anisotropy and of cyclic loading. However original Cam-clay is consistent with Coulomb's twice repeated[10] statement that the cohesion of newly remoulded soil is nil.

REFERENCES:

[1] Taylor, D.W. (1948), ***Fundamentals of Soil Mechanics,*** John Wiley & Sons.

[2] Hvorslev, M.J. (1937), ***Über die Festigkeitseigenschaften gestörter bindiger Böden*** (On the physical properties of disturbed cohesive soils), København.

[3] Roscoe, K.H., Schofield, A.N., and Wroth, C.P. (1958), "On the yielding of soils", ***Geotechnique,*** 8, 22-53.

[4] Roscoe, K.H. and Schofield, A.N. (1963), "Mechanical behaviour of an idealised. 'wet clay"', ***Proc. 2nd European Conf. on Soil Mechanics and Foundation Engineering, Wiesbaden,*** 1 , 47-54.

[5] Roscoe, K.H. and Burland, J.B. (1968), "On the generalised stress-strain behaviour of 'wet' clay", ***Engineering Plasticity***, edited by J. Heyman and F.A. Leckie, 535-609, Cambridge University Press.

[6] Muir Wood, D.M. (1990), ***Soil Behaviour and Critical State Soil Mechanics,*** Cambridge University Press.

[7] Shibata, T. (1963), "On the volume changes of normally-consolidated clays" (in Japanese), ***Disaster Prevention Research Institute Annuals, Kyoto University, 6,*** 128-34.

[8] Schofield, A.N. and Wroth C.P. (1968), ***Critical State Soil Mechanics,*** McGraw-Hill, London.

[9] Schofield, A.N. (1980), Cambridge Geotechnical Centrifuge operations, 20th Rankine Lecture, ***Geotechnique,*** 30(3), ***227-68***

[10] Heyman, J. (1972), ***Coulomb's Memoir on Statics: an essay in the History of Civil Engineering,*** Cambridge University Press.

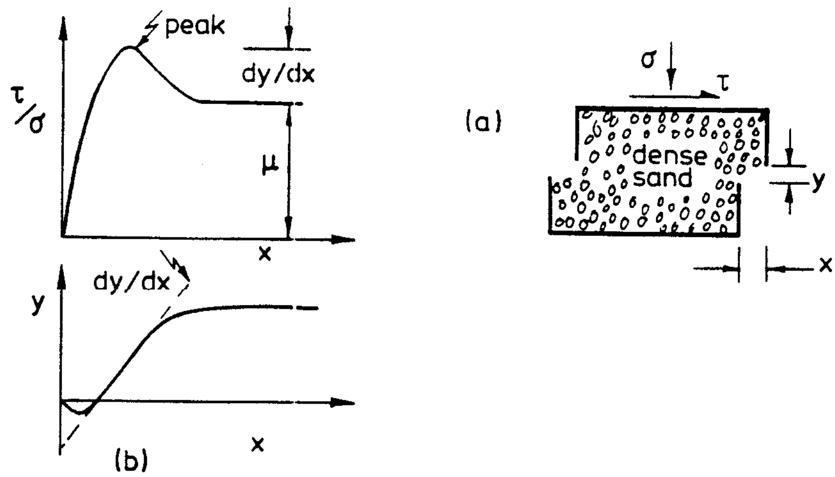


Fig. 1 Direct shear box tests of dense sand

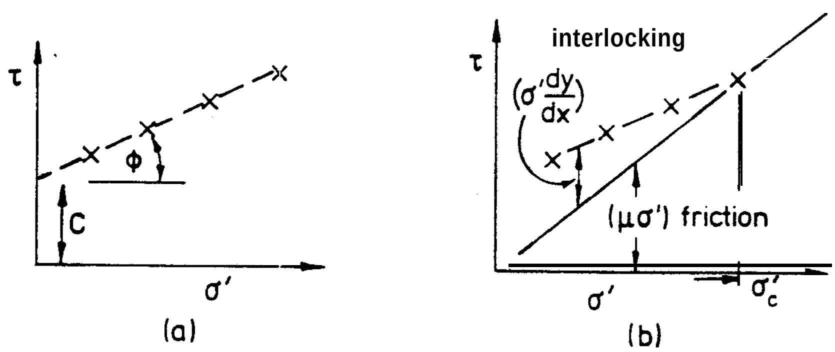


Fig. 2 Alternative interpretations

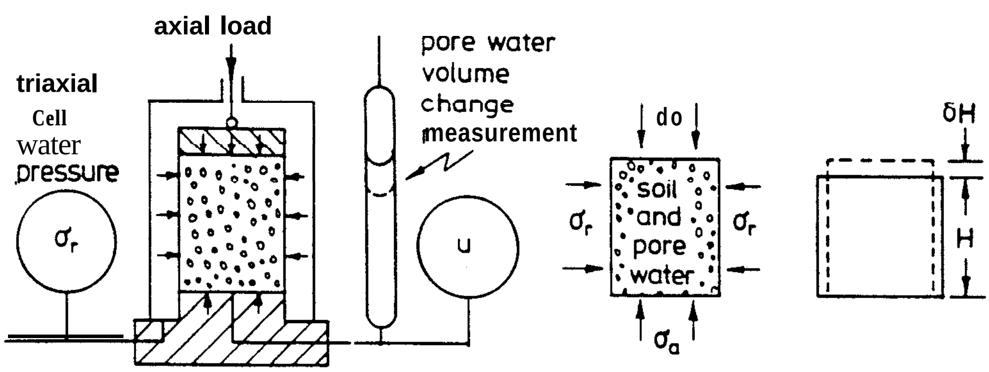


Fig. 3 The triaxial test

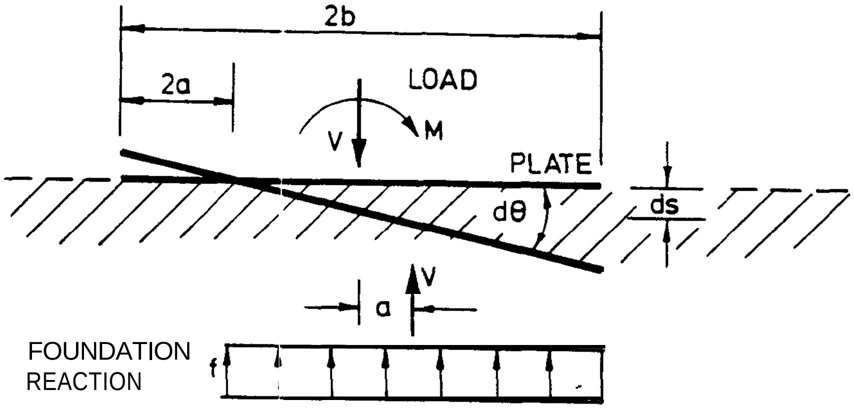


Fig. 4 A loaded plate

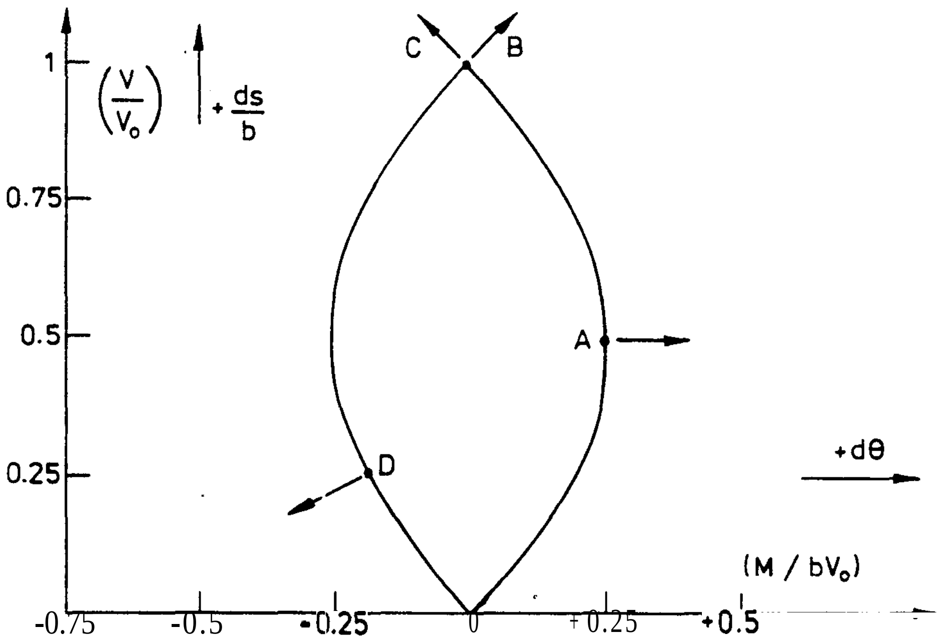


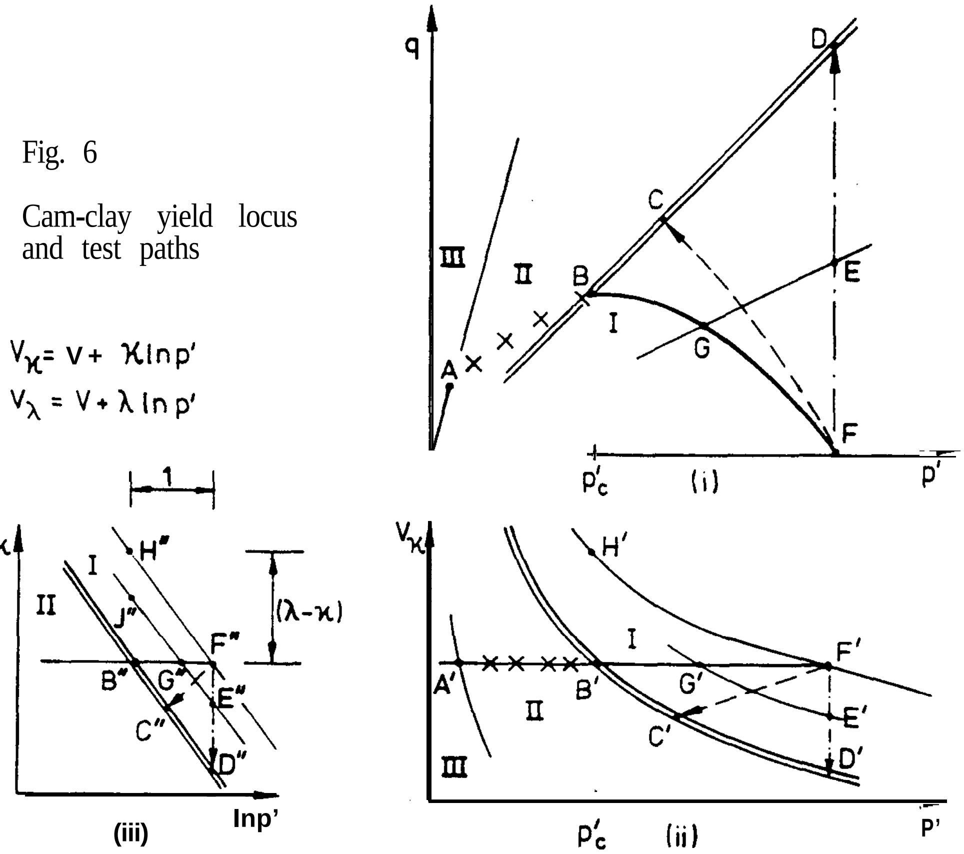
Fig. 5 Plate yield locus and associated flow

Fig. 6

Cam-clay yield locus and test paths

$V_{\kappa} = v + \kappa \ln p'$

$V_{\lambda} = V + \lambda \ln p'$



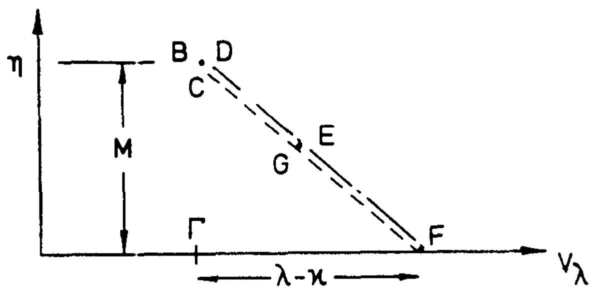


Fig. 7 Test paths for Cam-clay