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On the micromechanics of crushable aggregates

G. R. McDOWELL* and M. D. BOLTON*

This paper presents a study of the micromechanical behaviour of crushable soils. For a single grain loaded diametrically between flat platens, data are presented for the tensile strengths of particles of different size and mineralogy. These data are shown to be consistent with Weibull statistics of brittle fracture. Triaxial tests on different soils of equal relative density show that the dilatational component of internal angle of friction reduces logarithmically with mean effective stress normalized by grain tensile strength. The tensile strength of grains is also shown to govern normal compression. For a sample of uniform grains under uniaxial compression, the yield stress is related to the average grain tensile strength. If particles fracture such that the smallest particles are in geometrically self-similar configurations under increasing macroscopic stress, with a constant probability of fracture, a fractal geometry evolves with the successive fracture of the smallest grains, in agreement with the available data. A new work equation predicts that the evolution of a fractal geometry gives rise to a linear compression line when voids ratio is plotted against the logarithm of macroscopic stress, in agreement with published data.

KEYWORDS: compressibility; constitutive relations; plasticity; sands; statistical analysis.

Cet exposé présente une étude du comportement micro mécanique des sols concassables. Pour un seul grain chargé de manière diamétrale entre des platines plates, nous présentons les données de résistance à la rupture des particules de différentes dimensions et minéralogies. Nous montrons que ces données correspondent aux statistiques de Weibull sur la rupture de fragilité. Les essais triaxiaux sur divers sols de même densité relative montrent que le composant de dilatation de l'angle interne de friction baisse de manière logarithmique en même temps que la contrainte effective moyenne normalisée par la résistance à la rupture du grain. Nous montrons aussi que la résistance à la rupture des grains gouverne la compression normale. Pour un échantillon de grains uniformes sous compression uniaxiale, la limite élastique est liée à la rupture moyenne des grains. Si la rupture des particules est telle que les plus petites particules sont dans des configurations à similitude géométrique intrinsèque sous un effort macroscopique de plus en plus grand, avec probabilité constante de rupture, une géométrie fractale apparaît avec la rupture successive des grains les plus fins, en accord avec les données disponibles. Une nouvelle équation de travail prédit que l'évolution de la géometrie fractale provoque l'apparition d'une ligne de compression linéaire quand le taux de pores est représenté sous forme de courbe par rapport au logarithme de l'effort macroscopique, en accord avec les données publiées.

INTRODUCTION

It is well known that particle fracture plays a major role in the behaviour of crushable soils. For an aggregate of particles under normal compression, the yield stress of a calcareous sand is lower than that of a siliceous sand, simply because the calcareous particles are more friable (Golightly, 1990). The dilatational component of the internal angle of friction of a sand measured in a triaxial test is known to reduce logarithmically with mean effective stress (Bolton, 1986), since at high stresses, crushing eliminates the dilatancy. The normal compression of a sand is known to give rise to the evolution of a distribution of particle sizes (Fukumoto, 1992). In particular, it has been found that the particle size distributions of broken and crushed granular materials tend to be self-similar or fractal (Turcotte, 1986). The particle grading can be characterized by defining a fractal dimension which, remarkably, often tends to be about 2·5 for aggregates subjected to pure crushing (Turcotte, 1986; Palmer & Sanderson, 1991).

The detailed micromechanics of the phenomena mentioned above are not well understood. The

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tensile strength of particles must govern the behaviour of an aggregate, but how, for example, does normal compression give rise to a distribution of particle sizes? The difficulty begins with defining grain fracture, in order to obtain a consistent definition of particle strength.

This paper aims to relate the micromechanics of grain fracture to the macroscopic deformation of crushable aggregates. To this end, it is necessary first of all to obtain a reliable definition of grain fracture. Data from tests on grains crushed between flat platens are explained in terms of the Weibull statistics of fracture of brittle ceramics (Weibull, 1951), and this permits a consistent definition of grain tensile strength. The paper uses this definition to quantify the behaviour of various crushable soils. A study is made of the influence of grain strength on the internal angle of friction for dilatant, crushable aggregates. In addition, the micromechanics of normal compression are examined, with the specific purpose of explaining the existence of approximately linear compression lines in $e-\log\sigma'$ space. This leads to an expression for the compressibility index of the aggregate, in terms of more fundamental particle parameters.

FRACTURE OF A SINGLE PARTICLE

It is widely accepted that the failure of a spherical particle under compression is in fact a tensile failure. There is a tensile stress distribution induced in the particle and failure occurs when the tensile stress σ at a critical flaw of size a is such that the stress intensity factor $K = Y\sigma\sqrt{\pi a}$ (where Y is a dimensionless number relating to the problem geometry) for the flaw reaches the mode I fracture toughness of the material K_{Ic}, according to Griffith's criterion (Griffith, 1920). Consequently, the 'tensile strength' of rock grains can be indirectly measured by diametral compression between flat platens (Jaeger, 1967). Lee (1992) compressed individual grains of Leighton Buzzard sand, oolitic limestone and carboniferous limestone, in such a manner shown in Fig. 1. For a grain of diameter d under a diametral force F, a characteristic tensile stress induced within it may be defined as

$$\sigma = \frac{F}{d^2} \tag{1}$$

following Jaeger (1967) and Shipway & Hutchings (1993). This is also consistent with the definition of tensile strength of concrete in the Brazilian test. Fig. 2 shows a typical plot of platen load F as a function of the platen displacement δ. It can be seen that there are some initial peaks in the load–displacement curve, which correspond to the fracturing of asperities, and the rounding of the particle as small corners break off. For a particle of size d under load F, an asperity of size a may be considered to have an induced tensile stress of F/a^2, and the asperity would break off when this stress attained the critical value for the asperity. However, this cannot be described as failure of the particle: the asperity under the platen which has just broken off would have fractured irrespective of the size of the parent particle. Hence, a more reliable definition of particle breakage is required. It can be clearly seen in Fig. 2 that the initial small peaks are followed by a large peak corre-

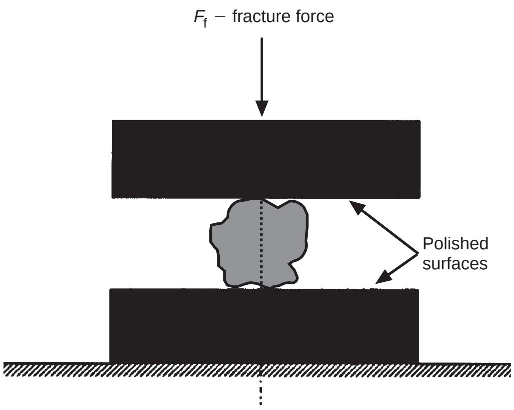

Fig. 1. Particle tensile strength test set-up (Lee, 1992)

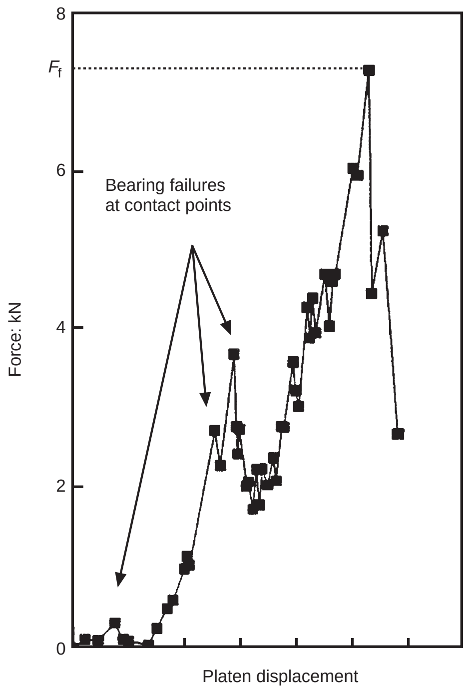

Fig 2. Typical load-deflection plot (Lee, 1992)

sponding to the maximum load, and a catastrophic failure as the particle splits, and the load drops dramatically. Particle fracture will henceforth be interpreted as particle 'splitting'. With this definition of fracture, Lee calculated the tensile strength of grains as

$$\sigma_{\mathrm{f}} = \frac{F_{\mathrm{f}}}{d^2} \tag{2}$$

where the subscript f denotes failure. He found that for particles of a given size and mineralogy, the tensile strength is not a constant but has a standard deviation about some mean value. Furthermore, he found the average tensile strength to be a function of particle size d. Fig. 3 shows the mean tensile strength σ_{f} as a function of the average particle size d. The data are described by the relation

$$\sigma_{\mathrm{f}} \propto d^b \tag{3}$$

where typical values of b are given by $-0{\cdot}357$, $-0{\cdot}343$ and $-0{\cdot}420$ for Leighton Buzzard sand, oolitic limestone and carboniferous limestone, respectively. This size effect on particle strength was also evident in particle crushing tests performed by Billam (1972), and it is a direct consequence of the statistical variation in the strength of brittle ceramics. Because ceramic materials contain a distribution of flaw sizes, small samples are stronger than large samples, since there are fewer and smaller flaws. This will now be quantified more rigorously using Weibull statistics of fracture, which is widely accepted to describe the tensile strength of brittle ceramics.

Weibull (1951) recognizes that the survival of a block of a material under tension requires that all its constituent parts remain intact (i.e. a chain is as strong as its weakest link). Weibull stated that for a volume V, under an applied tensile stress σ, the 'survival probability' $P_{\mathrm{s}}(V)$ of the block is given by

$$P_{\mathrm{s}}(V) = \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{4}$$

where V_0 is a reference volume of material such that

$$P_{\mathrm{s}}(V_0) = \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{5}$$

which is plotted in Fig. 4. The parameter σ_0 is the value of tensile stress σ such that 37% (i.e. $100\exp(-1)$%) of the total number of tested blocks survive. The exponent m is the Weibull modulus, and decreases with increasing variability in tensile strength. For chalk, stone, pottery and cement m is about 5. A similar value would be expected for carbonate sands, which have similar intraparticle porosity values. The engineering ceramics, such as Al_2O_3 have values for m of about 10, and the variation in strength is much less. For soils, we expect $5 < m < 10$. It can readily be shown that the mean tensile strength for samples of volume V_0 is given by

$$\sigma_{\mathrm{mean}} = \sigma_0 \Gamma(1 + 1/m) \tag{6}$$

where Γ is the gamma function, and returns a value of about 1 for the values of m being considered. It is also readily seen in Fig. 4 that the

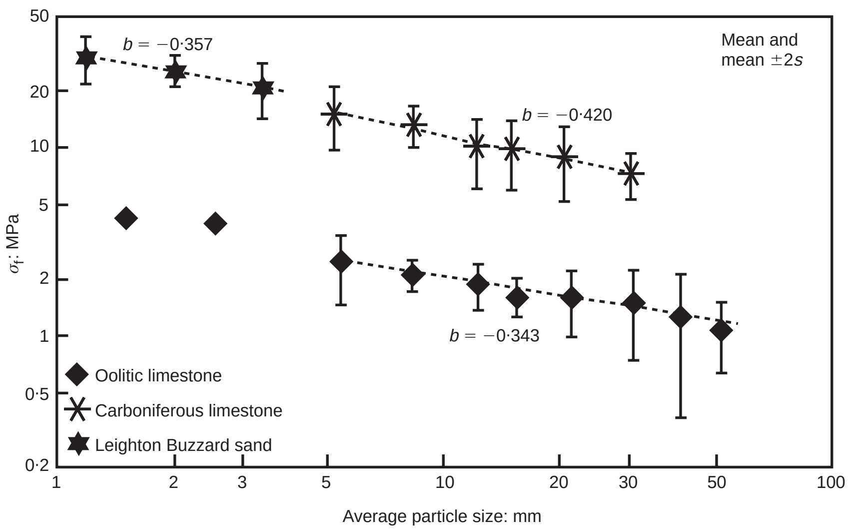

Fig. 3. Mean tensile strength as a function of particle size (Lee, 1992)

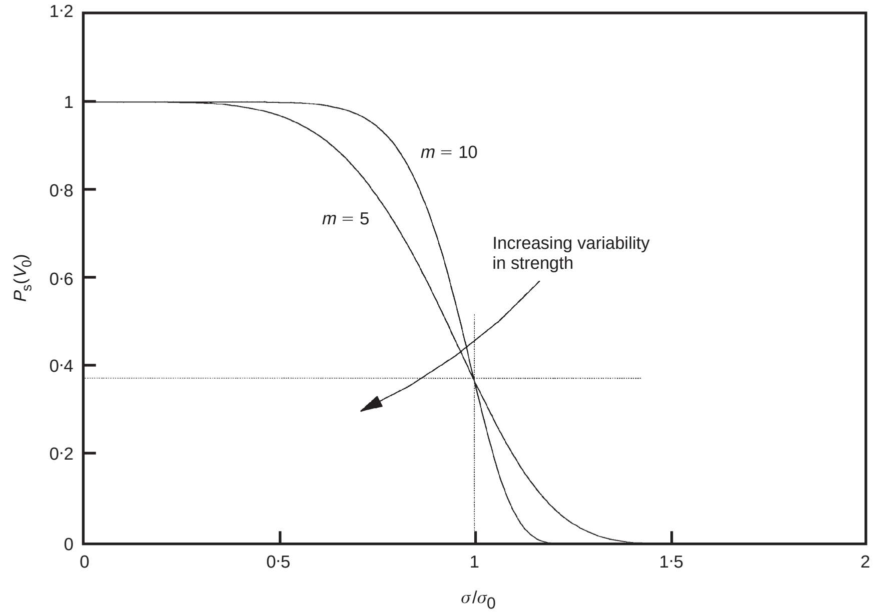

Fig. 4. Weibull distribution of strengths

maximum rate at which samples of volume V_0 fracture as the applied tensile stress increases occurs at around $\sigma = \sigma_0$. A more rigorous analysis shows that $dP_s/d\sigma$ is a minimum (i.e. $dP_f/d\sigma$ is a maximum, where P_f is the fracture probability) at a value of stress given by

$$\sigma = \sigma_0\left(\frac{m-1}{m}\right)^{1/m} \tag{7}$$

It is evident, then, that σ_0 is a significant and useful parameter in describing the strength of ceramic materials. For a block of material of volume V_1 under tension, equation (4) gives the 37% strength σ_{01} as

$$\sigma_{01} = \sigma_0\left(\frac{V_0}{V_1}\right)^{1/m} \tag{8}$$

and the mean strengths scale in the same way.

It is now necessary to examine the applicability of Weibull to the data presented by Lee in Fig. 3. If particles are approximately geometrically similar, and have the same number and distribution of contacts, the size of the zones of tensile stress induced within them must scale with their volume. In this case, Weibull applies. The survival probability of a particle size d under diametral compression is therefore given by

$$P_s(d) = \exp\left[-\left(\frac{d}{d_0}\right)^3\left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{9}$$

where σ is the characteristic tensile stress induced in the particle given by equation (1), and σ_0 is now the value of F/d^2 at which 37% of the tested particles survive, and is approximately equal to (and is proportional to) the mean tensile strength of particles of size d_0. It is evident from equations (8) and (9) that the average tensile strength of grains scales with particle size according to the relation

$$\sigma_0 \propto d^{-3/m} \tag{10}$$

which is equivalent to equation (3). Apparently, values of m in the range 5–10 cover Lee's data for rock grains in Fig. 3.

The above analysis concludes that the mean value of F/d^2 at fracture for grains compressed diametrically between flat platens is a proper statistical measure of the tensile strength of ceramic particles.

INFLUENCE OF GRAIN STRENGTH ON ANGLE OF INTERNAL FRICTION

Bolton (1986) proposed an empirical relation for the extra component of internal angle of friction $\Delta\phi$ due to dilatancy above the critical state strength ϕ'_{crit}, as a function of the mean effective stress p' and the initial relative density I_D:

$$\Delta\phi = \phi'_{max} - \phi'_{crit} = AI_R{}^\circ \tag{11}$$

where $A = 3$ for triaxial strain and $A = 5$ in plane strain conditions, and

$$I_R = I_D(Q - \ln p') - 1 \tag{12}$$

where p' is in kilopascals. The parameter Q relates to the mean effective stress required to suppress dilatancy, and takes a value of 10 for the quartz and feldspar sands discussed by Bolton. Bolton (1986) noted that triaxial tests by Billam (1972) had shown that reducing the crushing strength of the grains reduced the critical mean effective stress required to suppress dilatancy, implying that the value of Q in equation (12) should be reduced for soils of weaker grains. Bolton suggested values for Q of 8 for limestone, 7 for anthracite and 5·5 for chalk. However, since the average grain strength as measured by Lee (1992) has now been shown to be a proper statistical measure of tensile strength, it seems likely that the dilatational component of the internal angle of friction $\Delta\phi$ should be a function of the mean effective stress normalised by the average grain tensile strength σ_0 for soils of equal relative density.

Figure 5 shows results of triaxial tests performed by Lee (1992) on samples of Leighton Buzzard sand and oolitic limestone at the same initial relative density. The results include two different particle size gradings for each soil: samples contained approximately uniformly sized particles, but the size of the grains, varied by a factor of about 10 between the two gradings. The figure shows the dilatational component of angle of friction $\Delta\phi$ plotted against p'/σ_0 on a logarithmic scale. It is clear that $\Delta\phi$ reduces linearly with $\log(p'/\sigma_0)$, thus confirming the hypothesis that the tensile strength of grains should govern the dilatant behaviour of crushable soils. More micromechanical insight can therefore be achieved if equation (12) is rewritten as

$$I_R = I_D \ln\left(\frac{B\sigma_0}{p'}\right) - 1 \tag{13}$$

where B is a scalar multiplier. As Bolton (1986) remarked, this expression is applicable for $I_R \geqslant 0$, and should not be taken to apply as p' approaches $B\sigma_0$. The onset of contraction is the subject of the next section.

ONE-DIMENSIONAL COMPRESSION

There now follows a study of the micromechanics of one-dimensional compression. In particular, the aim is to determine the role of grain fracture in the uniaxial compression of crushable soils. Fig. 6 (Golightly, 1990) shows typical plots of voids ratio against the logarithm of vertical effective stress for samples of carbonate and silica sands which have been compressed in an oedometer. Plots for loose and dense silica sand are shown for comparison. Consider the compression of dense silica sand. A geotechnical engineer might describe the form of the curve by the relationship

$$e = f(\sigma'_v) \tag{14}$$

which is dimensionally inconsistent: the vertical effective stress ought to be normalized by some material parameter with the dimensions of stress. Bolton & McDowell (1996) suggested that for the small deformations in region 1, the normalizing

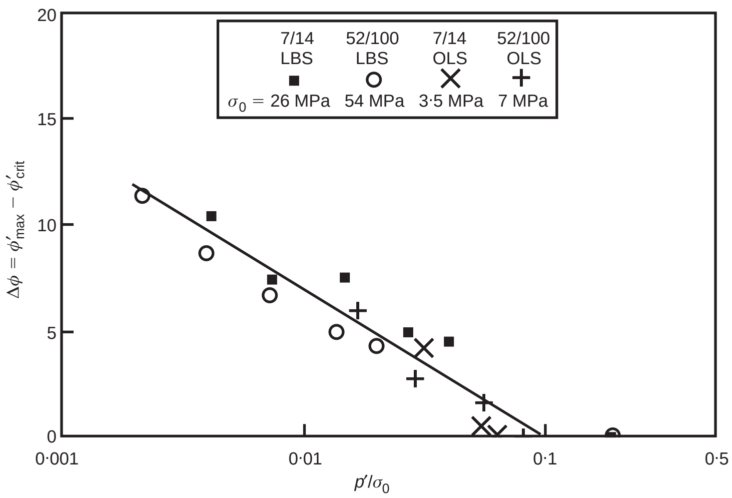

Fig. 5. Dilatational component of angle of internal friction plotted against mean effective stress normalized by grain tensile strength (Lee, 1992). LBS, Leighton Buzzard sand; OLS, oolitic limestone

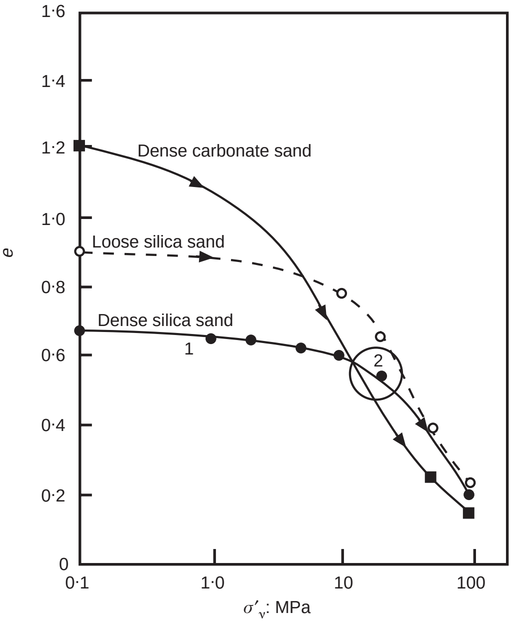

Fig. 6. One-dimensional compression plots for carbonate and silica sands (Golightly, 1990)

parameter should be the elastic modulus of a particle, so that one might write

$$e = f(\sigma'_{\mathrm{v}}/G_{\mathrm{p}},\ \phi) \tag{15}$$

where ϕ is the angle of internal friction. Even with a dense sand, small irrecoverable deformations may occur by particle rearrangement, since grains not in their closest possible packing may still rearrange. This explains why the relative density of a vibro-compacted sand increases with increasing time spent on vibro-compaction.

It is evident in Fig. 6 that a well-defined yielding region 2 exists. This cannot be due to particle rearrangement alone: for this dense sand, which has exhausted all possible particle rearrangement at region 2, particle breakage is a prerequisite for further compaction. For an array of approximately uniform particles under compression, the average characteristic tensile stress σ induced in a grain must be proportional to the applied macroscopic stress $\overline{\sigma}$ (where the overbar will now be used to indicate macroscopic, compressive). For particles loaded in this manner, there will exist a value of characteristic induced tensile stress σ_0 such that 37% of particles survive. As previously noted, σ_0 also represents (approximately) the characteristic tensile stress at which the rate of particle fracture with increasing stress $\mathrm{d}P_{\mathrm{f}}/\mathrm{d}\sigma$ is a maximum, which we may consider to correspond to the yielding region 2. The yield stress, then, must be proportional to the average tensile strength of grains, as measured by crushing between flat platens, so that for irrecoverable strains in region 2 we may write

$$e = \mathrm{f}(\overline{\sigma}/G_{\mathrm{p}},\ \phi,\ \overline{\sigma}/\sigma_0) \tag{16}$$

Bolton & McDowell (1996) called the behaviour in region 2 'clastic' yielding, whereby major irrecoverable deformations are permitted by the onset of particle fracture. A clastic yield stress $\overline{\sigma}_0$ may be defined as the value of macroscopic stress which causes the maximum rate of grain fracture under increasing stress, for particles loaded in this configuration. For an aggregate of brittle particles of very high Weibull modulus (i.e. a unique particle tensile strength), it is evident that catastrophic compression of the aggregate would occur when the applied macroscopic stress attained a value equal to the clastic yield stress. McDowell (1997) showed, using a simple numerical model, that this is indeed the case.

The compression plot in Fig. 6 for loose silica sand shows that at higher voids ratios more particle rearrangement can occur before the onset of crushing, in agreement with data published by Hagerty *et al.* (1993). Furthermore, the clastic yield stress is lower, since at higher voids ratios the average coordination number (number of contacts per particle) in the aggregate reduces (Oda, 1977). Jaeger (1967) showed that for circular particles under combinations of surface forces, the maximum tensile stress in a particle reduces as the coordination number increases.

The compression plot for the dense carbonate sand also shows a yielding region, although it is less pronounced than that for the dense silica sand. This should be expected from a material with a lower Weibull modulus, which is what one might anticipate for a calcareous sand which is composed of very porous grains.

'NORMAL' COMPRESSION

Figure 6 shows that for a crushable aggregate which has been one-dimensionally compressed, clastic yielding is followed by a region of plastic hardening, in which the macroscopic stress must be increased in order to produce any further compaction. Geotechnical engineers call this 'normal' compression. Furthermore, there is a linear relationship between voids ratio and the logarithm of effective stress. From early publications in soil mechanics, the one-dimensional plastic compression of soils has been shown to satisfy this relationship. The resulting plots are known as 'normal compression lines', and the slope of such a plot, the compressibility index, is taken to be an empirical soil constant, given the symbol λ when using natural logarithms, in Cam Clay (Schofield & Wroth, 1968), and subsequent models based on plasticity theory (Roscoe & Burland, 1968; Miura *et al.,* 1984):

$$e = e_0 - \lambda \ln \overline{\sigma} \quad (17)$$

Terzaghi (1948) used a similar equation with common logarithms, and the slope was given the symbol C_c. Equation (17), like equation (14), is dimensionally inconsistent and physically incomplete—the macroscopic stress should be normalized by a material parameter with dimensions of stress. The popular use of atmospheric pressure, or a standard stress (1 kPa, 1 kgf/cm^2, etc.) is, of course, no answer to this dilemma. The principles of physics demand that behaviour be described in terms which are dimensionless with respect to parameters involved in the physical process. Furthermore, the physical origins of λ have remained a mystery to geotechnical engineers: it has seemed impossible to formulate a non-dimensional group for λ in terms of particle parameters. The linear–log relationship described by equation (17) applies to a wide range of geotechnical materials (Novello & Johnston, 1989), and the compressibility index λ is remarkably in the range 0·1–0·4 for a wide range of granular materials.

It is well known that the normal compression of crushable soils leads to particle size disparity (Fukumoto, 1992; Hagerty *et al*., 1993). The aim of this paper is now to relate the evolution of a distribution of particle sizes to the existence of the normal compression lines evident in Fig. 6.

THE EXISTENCE OF FRACTALS

It has long been recognized that a wide variety of scale-invariant processes (magnification does not alter the picture) occur in nature. The concept of fractals provides a means of quantifying these processes (Mandelbrot, 1982). Turcotte (1986) examined the fragment sizes for a wide range of crushed materials and found the size distributions to have a fractal character. A fractal defines a simple power law relation between the number of particles and their size, so that the number of fragments which have a diameter (or other characteristic linear dimension) size L greater than size d is given by

$$N(L > d) = Ad^{-D} \quad (18)$$

where A is a constant of proportionality and D is the fractal dimension. The fractal dimension D is often about 2·5 for materials subjected to pure crushing—for example Turcotte (1986) gives broken coal a value of $D = 2{\cdot}50$; granite fragments from an underground nuclear explosion have $D = 2{\cdot}50$; and basalt fragments from projectile impact have $D = 2{\cdot}56$. The materials which exhibit fractal dimensions significantly different from 2·5 are usually not the result of pure crushing, and have often been subject to sorting—for example, glacial till has an observed fractal dimension of 2·88 (Turcotte, 1986). It seems likely that there is a mechanical explanation for the existence of a fractal dimension of about 2·5—Palmer & Sanderson (1991) produced a simple fractal crushing model for ice, and showed that if the crushing force on a block of ice is determined equally by the fracture of fragments of all sizes (i.e. all fragment sizes make an equal contribution to the crushing force), then the fractal dimension must be 2·5, in agreement with observed values of D. Steacy & Sammis (1991) used computer simulations to examine the evolution of a fractal geometry under different conditions, in order to explain the observed fractal dimensions in fault gouge of $2{\cdot}6 \pm 0{\cdot}1$ (Sammis *et al*., 1987). Their model consists of a cube of material which splits into self-similar cubes, such that no neighbours are of the same size at any scale, since this would maximize the stress concentrations on a particle surface. Steacy & Sammis showed that if neighbours are defined as blocks sharing faces or edges, then it is possible to generate arrays of particles with a fractal dimension of about 2·5.

It will therefore be assumed here for simplicity that normal compression gives rise to a fractal distribution of particles with a value of $D = 2{\cdot}5$. McDowell *et al*. (1996) have given a more general analysis of aggregate compaction, in which the fractal dimension need not be 2·5; this will be discussed briefly later. It is essential at the outset to recognize that although a strictly fractal distribution of particle sizes obeying equation (18) would extend over an infinite range of scales, the fractal distribution must, in reality, be limited (Fig. 7).

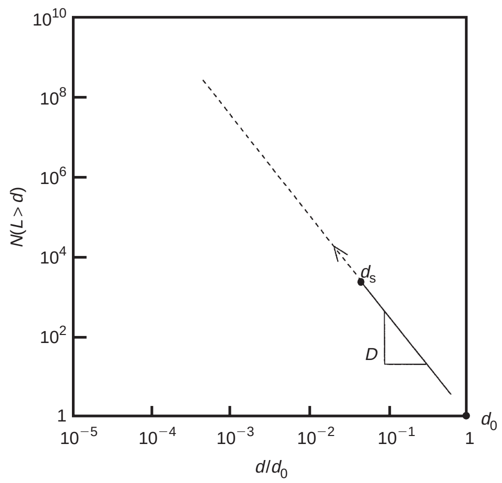

Fig. 7. A limited fractal of dimension *D*. The smallest particle size reduces under increasing stress

There will be a 'largest' particle of size d_0 corresponding to the original population, and a smallest particle of size d_s which will generally reduce as stress increases. This is of uttermost importance in understanding the micromechanics of normal compression. If the size distribution in Fig. 7 were strictly a fractal without limit, the aggregate would have a voids ratio uniquely defined by the fractal dimension D: scale invariance would demand it and make the process of normal compression impossible. The model for 'plastic' or 'clastic' irrecoverable compression which will now be presented is wholly dependent on the reduction of the smallest particle size d_s by successive splitting, so that voids ratio reduces as the broken fragments fill pre-existing voids.

FRACTAL CRUSHING

The probability of fracture of a particle is determined (McDowell *et al*., 1996) by the applied macroscopic stress, the size of the particle and the coordination number (number of contacts with neighbouring particles). The fracture probability $P_f(d)$ must increase with any increase in macroscopic stress $\overline{\sigma}$, but reduce with a decrease in particle size, or an increase in the coordination number (since an increase in the number of contacts must reduce the induced tensile stresses (Jaeger, 1967)). For an aggregate with particle size disparity, the largest grains will have very high-coordination numbers, since they will be surrounded by many smaller particles. The smallest particles in the aggregate must have the minimum coordination.

It might be anticipated, at first sight, that because small particles are stronger than large particles the largest particles are always the most likely to fracture. This would lead to the evolution of a uniform matrix of fine particles from the compression of an aggregate of coarse grains, behaviour which is not evident in the geotechnical literature. However, although the smallest particles are the strongest, they also have the fewest contacts. There are two opposing effects on particle survival, size and coordination number. If coordination number dominates over particle size in the evolution of the aggregate, then the smallest particles always have the highest probability of fracture. In this case, the compression of an aggregate of uniform grains would lead to a disparity in particle sizes, in which a proportion of the original grains is retained under the protection of a uniform compressive boundary stress created by its many neighbours. This type of behaviour is evident in Fig. 8, which shows the particle size distribution curve which evolves for Ottawa sand. McDowell *et al*. (1996) have shown, using a simple numerical model, that when coordination number dominates over particle size in determining the fracture probability of a particle, a fractal distribution of particle sizes evolves.

Suppose that it is the smallest particles which continue to fracture under increasing macroscopic stress, and that the probability of fracture of the smallest particles is a constant p, and that the number of fragments produced when a particle splits is a constant n. The scale-invariant parameters p and n will be shown to define a fragment size distribution which is a fractal. Consider a hierarchy of splitting grains such that the largest particles are size d_0, and that subsequent 'orders' of particle size are d_1, d_2, ..., d_s, where d_s is the smallest particle size which decreases under increasing macroscopic stress. If there were origin-

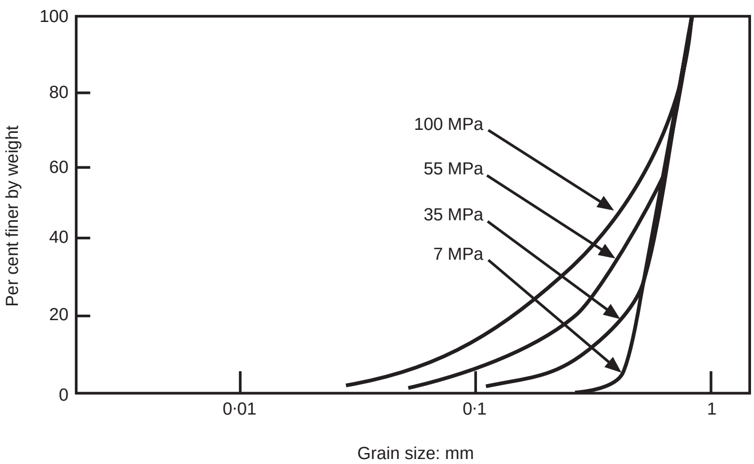

Fig. 8. Evolving particle size distribution curves for one-dimensionally compressed Ottawa sand (Fukumoto, 1992)

ally ω particles of order 0, then the number of particles of order i must be $\omega(np)^i(1-p)$. The number of particles of size d_i or greater is given by

$$N(L > d_i) = \omega(np)^i(1-p)[1 + 1/np + 1/(np)^2 + \ldots] \quad (19)$$

which reduces to

$$N(L > d_i) = \omega(np)^i(1-p)[1 - 1/np]^{-1} \quad (20)$$

for $np > 1$. The number of fragments of size d_{i+1} or greater is given by replacing i by $i+1$ in equation (20), so that the ratio of the number of fragments of order i or greater, to the number of fragments of order $i+1$ or greater is easily calculated as

$$\frac{N(L > d_i)}{N(L > d_{i+1})} = \frac{1}{np} \quad (21)$$

This defines a fractal distribution. Equation (18) gives

$$\frac{N(L > d_i)}{N(L > d_{i+1})} = \left(\frac{d_i}{d_{i+1}}\right)^{-D} = (n^{1/3})^{-D} \quad (22)$$

so that by comparing equations (21) and (22), the fractal dimension can be related to p and n:

$$D = 3\left(1 + \frac{\ln p}{\ln n}\right) \quad (23)$$

The idea of having a probability of fracture which is constant for the smallest particles is consistent with the idea that the smallest particles remain in geometrically similar configurations, with a minimum coordination. In this case, it is also evident that the smallest grains will, on average, have a characteristic induced tensile stress which is proportional to the current macroscopic stress. Microscopically, strong columns of force may form at intervals of a few particle diameters to transmit the major principal stress (Cundall & Strack, 1979). However, the organization of particles must change when any particle fractures, so that the snapshot of order at any instant should be irrelevant. The tensile stress induced in the smallest particles can be considered to take some characteristic value when integrated over many fractures.

We now proceed to relate the smallest particle size d_s to the current macroscopic stress $\overline{\sigma}$. Consider a material containing Griffith flaws (Griffith, 1920), and for which the maximum flaw size in a particle is proportional to the size of the particle (i.e. there is statistical self-similarity in the flaw size distribution), which corresponds to a disordered granular material. Following Carpinteri (1994), such a material would exhibit Griffith's law of fracture, with a size index in equation (3) of $b = -0{\cdot}5$. Since the smallest particles of size d_s may be considered to have a characteristic tensile stress which is proportional to the macroscopic stress $\overline{\sigma}$, and if the splitting of grains is governed by linear elastic fracture mechanics so that the only material parameter involved is the fracture toughness K_{Ic}, then the only dimensionless group that can be formed from these parameters is $\overline{\sigma}\sqrt{d_s}/K_{Ic}$—which must remain constant for geometrically similar situations, that is,

$$\overline{\sigma}\sqrt{d_s} \propto K_{Ic} \quad (24)$$

Alternatively, we may write

$$\overline{\sigma}\sqrt{d_s} \propto \overline{\sigma}_0\sqrt{d_0} \quad (25)$$

where $\overline{\sigma}_0$ is the clastic yield stress for the aggregate, and the constant of proportionality in equation (25) may be derived by equating the fractal probability of fracture p to the Weibull fracture probability P_f (McDowell, 1997), and may be taken to be about 1. Equation (25) relates the current smallest particle size to the applied uniaxial compressive stress for a material which obeys Griffith's laws of linear elastic fracture mechanics, and provides a mechanism for plastic hardening of the aggregate. Bolton & McDowell (1996) have termed this behaviour 'clastic hardening'. It should be noted that granular materials having flaw size distributions other than that which is described above have been treated by McDowell *et al.* (1996) using Weibull statistics, and have been omitted here for clarity.

CLASTIC HARDENING

It is now proposed to relate the evolution of a fractal geometry to the evolution of a 'normal compression line' which is linear in $e-\log\overline{\sigma}$ space. For this purpose, a new work equation will be introduced. It is first of all useful, however, to consider the original Cam Clay work equation (Roscoe *et al.*, 1963; Schofield & Wroth, 1968), which has been used very widely to model the deformation of a porous aggregate:

$$q\,\delta\varepsilon_q^p + p'\,\delta\varepsilon_v^p = Mp'\delta\varepsilon_q^p \quad (26)$$

The left-hand side represents the plastic work done per unit volume by deviatoric stress q and mean effective stress p' (with corresponding irrecoverable strain increments $\delta\varepsilon_q^p$ and $\delta\varepsilon_v^p$). The right-hand side was identified as internal frictional dissipation. If elastic strains are ignored, then the equation reduces to the Granta Gravel work equation (Schofield & Wroth, 1968), which will be used here for simplicity: elastic strains may easily be superimposed (McDowell *et al.*, 1996). We now add a term on the right-hand side of equation (26) for the energy dissipated in the successive fracture of brittle particles and get (McDowell *et al.*, 1996):

$$q\,\delta\varepsilon_q + p'\,\delta\varepsilon_v = Mp'\,\delta\varepsilon_q + \frac{\Gamma\,dS}{V_s(1+e)} \tag{27}$$

where dS is the increase in surface area of a volume V_s of solids distributed in a gross volume $V_s(1+e)$, and Γ is the 'surface energy', related to the critical strain energy release rate G_c by $\Gamma = G_c/2$ (not to be confused with the gamma function in equation (6)). For the special case of one-dimensional normal compression with effective axial stress $\overline{\sigma}$ and uniaxial strain $\overline{\varepsilon}$, we obtain

$$\overline{\sigma}\,d\overline{\varepsilon} = \tfrac{2}{9}M(1+2K_0)\overline{\sigma}\,d\overline{\varepsilon} + \frac{\Gamma\,dS}{V_s(1+e)} \tag{28}$$

where K_0 is the lateral/axial effective stress ratio, and may readily be estimated using Jâky's formula (Jâky, 1944) $K_0 \approx 1 - \sin\phi$. Substituting

$$d\overline{\varepsilon} = -\frac{de}{(1+e)} \tag{29}$$

into equation (28) relates the reduction in voids ratio to the increase in surface area in the soil sample:

$$de = -\frac{\Gamma\,dS}{(1-\mu)\overline{\sigma}V_s} \tag{30}$$

where μ is a weak function solely of the angle of internal friction (McDowell, 1997), varying from 0·4 for $\phi = 20°$ to 0·6 for $\phi = 40°$, and so may conveniently be assumed to take a value of 0·5. It is worth noting that equation (27) removes a limitation of the original Cam Clay work equation (Schofield & Wroth, 1968) by permitting plastic volume changes for simple isotropic loading. This new work equation predicts that under isotropic loading (i.e. $q = 0$, $d\varepsilon_q = 0$), an increase in the surface area of particles produces a finite volumetric strain, so that $d\varepsilon_v/d\varepsilon_q$ is infinite. In this case, if the normality rule were to be applied in order to derive a yield surface, the new work equation (27) would tend to round off the corner of the Cam Clay yield locus.

The total surface area of particles in the samples can be found by consideration of equation (18). By differentiating equation (18), it is evident that the number of particles with a size in the range of d to $d + \delta d$ is

$$\delta N = ADd^{-D-1}\,\delta d \tag{31}$$

The surface area of a particle of size d may be given by

$$S(d) = \beta_s d^2 \tag{32}$$

where β_s is the surface shape factor, so that the total surface area of particles of size in the range d to $d + \delta d$ is

$$\delta S = \beta_s ADd^{1-D}\,\delta d \tag{33}$$

The total surface area of particles in the sample $S(L > d_s)$ is given by integrating equation (33):

$$S(L > d_s) = 5\beta_s A d_s^{-1/2} \tag{34}$$

using $D = 2{\cdot}5$. Incidentally, the mass distribution may be calculated in the same way, so that the volume of particles less than or equal to size d is given by

$$M(L < d) = 5\beta_v A d^{1/2} \tag{35}$$

where β_v is the volume shape factor for the material. This is the final mass distribution which evolves during the successive fracture of the smallest grains. The total volume of solids in the sample V_s is given by equation (35) with $d = d_0$. Combining equations (34) and (25) relates the total surface area in the sample to the current applied macroscopic stress:

$$S = 5\beta_s A\left(\frac{\overline{\sigma}}{\overline{\sigma}_0}\right)\frac{1}{\sqrt{d_0}} \tag{36}$$

Combining equation (36) with equation (30) provides a clastic hardening law:

$$de = -\left(\frac{1}{1-\mu}\frac{\beta_s}{\beta_v}\frac{\Gamma}{\overline{\sigma}_0 d_0}\right)\frac{d\overline{\sigma}}{\overline{\sigma}} \tag{37}$$

which is a normal compression line, of slope

$$\lambda = \frac{1}{1-\mu}\frac{\beta_s}{\beta_v}\frac{\Gamma}{\overline{\sigma}_0 d_0} \tag{38}$$

It is evident, then that the evolution of a fractal geometry provides a micromechanical commentary for the evolution of normal compression lines. It is interesting to substitute some typical values for the parameters in equation (38). For a quartz sand, with $\phi \approx 30°$, $\mu \approx 0{\cdot}5$, Harr (1977) quotes values of β_s/β_v for crushed quartz in the range 14–18 (where the shape factors are defined using the projected diameter of a particle), so we might assume a value 16; Ashby & Jones (1986) give values of surface energy Γ for rocks of 25 J/m^2; and the clastic yield stress for a silica sand comprising 1 mm diameter particles might be 10 MPa. These values give a compressibility index $\lambda \approx 0{\cdot}1$ in equation (38), which is of the correct order of magnitude, comparing with available data for granular materials (Novello & Johnston, 1989).

DISCUSSION

The value of compressibility index $\lambda \approx 0{\cdot}1$ given by equation (38) must be partly fortunate: the least certain parameter is the surface energy Γ, which is difficult to estimate. Furthermore, the dependence of λ on d_0 is surprising, although there is a lack of experimental data to suggest that the compressibility index should be independent of the initial particle size. One possible source of explanation is the surface energy Γ. The surface energy has traditionally been taken to be a material constant (Ashby & Jones, 1986), related to the mode I fracture toughness K_{Ic} by $K_{Ic} = \sqrt{2E\Gamma}$ (where E is Young's modulus) when the principles of linear elastic fracture mechanics apply. However, recent tensile testing experiments on concrete (Carpinteri, 1994), and tensile splitting experiments on short rods of sandstone and marble (Scavia, 1996) have shown that over a range of scales for these regular geometries,

$$\Gamma \propto d^{D_s - 2} \tag{39}$$

where d is some characteristic length of the specimen and D_s is the surface fractal dimension, with $2 < D_s < 3$. In fact, the kinematics of the fracture process puts an upper limit on D_s of 2·5, which corresponds to an extremely disordered material, with a wide distribution of flaws. For a material containing Griffith flaws, with the maximum flaw size proportional to the size of the specimen, then $\Gamma \propto d^{1/2}$. If we now rewrite equation (38) as

$$\lambda = \frac{1}{1-\mu}\frac{\beta_s}{\beta_v}\frac{\Gamma/\sqrt{d_0}}{\overline{\sigma}_0\sqrt{d_0}} \tag{40}$$

then $\overline{\sigma}_0\sqrt{d_0} \propto K_{Ic}$, which is a material constant (Xie, 1993), so that in the application of linear elastic fracture mechanics it might seem plausible to scale the surface energy in proportion to $d_0^{1/2}$, so that λ is seen to be a material constant. Nevertheless, what is important is that equation (38) predicts a unique value of λ for a particular aggregate, so that if a normally compressed soil is unloaded and reloaded, the normal compression line will be rejoined at the preconsolidation stress. The current preconsolidation or yield stress is determined by the tensile strength of the smallest grains. If the smallest particle size is known, then the current voids ratio may be uniquely defined by the relationship

$$e = f(\overline{\sigma}/G_p, \phi, \overline{\sigma}/\sigma_{0s}) \tag{41}$$

where σ_{0s} is the tensile strength of the smallest particles.

The derivation of the e–$\log\overline{\sigma}$ relationship in this paper has been for disordered materials in which the average size of a critical flaw is proportional to the size of a particle, and which form particle size distributions with a fractal dimension of 2·5. A more general analysis has been given by McDowell (1997), which shows that the voids ratio–effective stress relationship is given by

$$\mathrm{d}e = -\lambda(\overline{\sigma})^{(m/3)(D-2)-1}\frac{\mathrm{d}\overline{\sigma}}{\overline{\sigma}} \tag{42}$$

so that for the majority of granular materials with $D \approx 2·5$, and $5 < m < 10$, the power on stress in equation (42) is negligible, which explains why most normal compression curves are approximately linear in e–$\log\overline{\sigma}$ space. However, the purpose of this paper has been to elucidate the fundamental micromechanics of normal compression and provide a mechanism of plastic (clastic) hardening.

There is a lack of experimental data to show that the existence of linear normal compression lines for crushable soils is consistent with the evolution of a fractal geometry. However, the theory is supported by the data in Fig. 9, which shows a normal compression curve for petroleum coke, together with the particle size distribution which

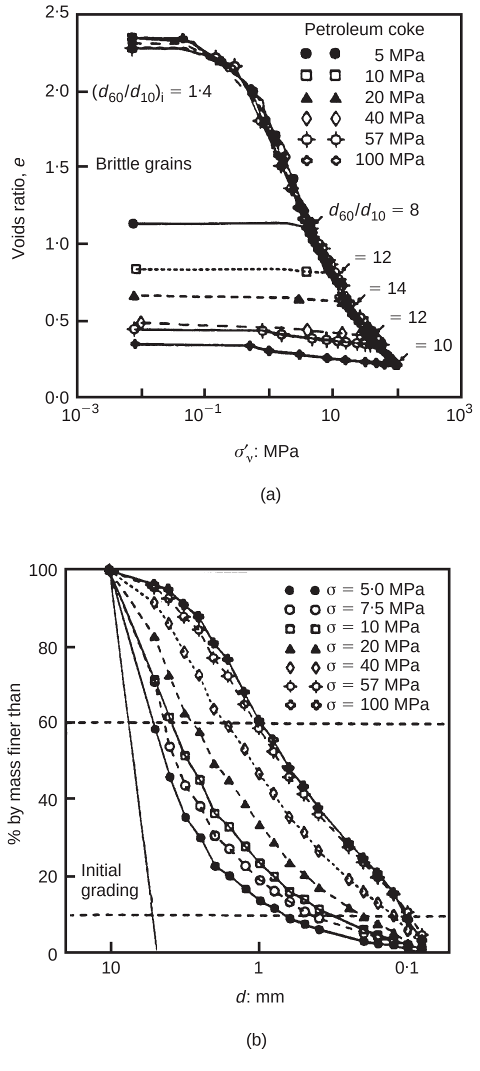

Fig. 9. (a) One-dimensional normal compression curve for petroleum coke. (b) Evolving particle size distributions for one-dimensionally compressed petroleum coke

evolves (Bard, 1993; Biarez & Hicher, 1994). It is clear that the evolution of the linear portion of the curve in Fig. 9(a) is consistent with the evolution of a fractal distribution of particle sizes in Fig. 9(b) as the uniformity coefficient tends to some constant value. The change in curvature of the plot at low voids ratios must be due to the comminution limit (Kendall, 1978) for petroleum coke— Fig. 9(b) (Bard, 1993) shows that at high stresses there is significant breakage of the larger grains, since the smallest particles are so small that they yield before fracture. However, the data clearly support the hypothesis that the micromechanical origin of normal compression lies in the evolution of a fractal geometry. It seems that 'normal compression' would be better termed 'fractal compression'.

CONCLUSIONS

The tensile strengths of soil grains compressed between flat platens satisfy the Weibull statistics of fracture of brittle ceramics, which permits a useful and consistent definition of grain strength. This grain strength governs the strength and dilatancy of crushable soils, so that at a given relative density the dilatational component of the angle of internal friction is proportional to the logarithm of mean effective stress normalized by grain tensile strength. For an aggregate under one-dimensional normal compression, the yield stress is proportional to the grain tensile strength. The successive fracture of the smallest particles under increasing macroscopic stress to form a limited fractal geometry provides a micromechanical insight into the existence of linear 'normal compression lines'. In this case, the current yield stress of the aggregate is determined by the tensile strength of the smallest particles.

NOTATION

b	Lee index on tensile strength
d	particle size
d_0	size of largest particle, order 0
d_i	size of a particle of order i
d_s	size of smallest particle, order s
D	fractal dimension
e	voids ratio
F	diametral force for a particle compressed between flat platens
F_f	crushing force for a particle compressed between flat platens
G_p	shear modulus of particle material
I_D	relative density
I_R	relative dilatancy index
K_{Ic}	mode I fracture toughness
K_0	lateral/axial effective stress ratio in one-dimensional compression
m	Weibull modulus
$M(L<d)$	volume of particles finer than size d
n	number of fragments produced when a particle splits
$N(L>d)$	number of particles larger than size d
p	fractal probability of fracture
p'	mean effective stress
$P_f(d)$	fracture probability for a particle of size d
$P_s(d)$	survival probability for a particle of size d
$P_s(V)$	survival probability for a volume V
q	deviatoric stress
Q	Bolton constant
S	surface area
$S(L>d)$	total surface area of particles larger than size d
V	volume
V_0	reference volume
V_s	volume of solids
β_s	surface shape factor
β_v	volume shape factor
ε_q	triaxial shear strain
ε_v	volumetric strain
$\overline{\varepsilon}$	uniaxial macroscopic compressive strain
ϕ	angle of friction
Γ	surface energy
λ	compressibility index
μ	frictional constant for one-dimensional normal compression
M	critical state frictional dissipation constant
σ	characteristic tensile stress induced in a particle
σ_f	Lee tensile strength of a particle
σ_0	Weibull 37% tensile strength
σ'	effective stress
σ'_v	vertical effective stress
$\overline{\sigma}$	uniaxial macroscopic compressive stress
$\overline{\sigma}_0$	clastic yield stress

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