McMahon, B. T. *et al.* (2013). *Géotechnique* **63**, No. 16, 1447–1450 [http://dx.doi.org/10.1680/geot.13.T.002]

TECHNICAL NOTE

Optimal displacement mechanisms beneath shallow foundations on linear-elastic perfectly plastic soil

B. T. MCMAHON*, S. K. HAIGH* and M. D. BOLTON*

An energy method for a linear-elastic perfectly plastic method utilising the von Mises yield criterion with associated flow developed in 2013 by McMahon and co-workers is used to compare the ellipsoidal cavity-expansion mechanism, from the same work, and the displacement fields of other research by Levin, in 1995, and Osman and Bolton, in 2005, which utilise the Hill and Prandtl mechanisms respectively. The energy method was also used with a mechanism produced by performing a linear-elastic finite-element analysis in Abaqus. At small values of settlement and soil rigidity the elastic mechanism provides the lowest upper-bound solution, and matches well with finite-element analysis results published in the literature. At typical footing working loads and settlements the cavity-expansion mechanism produces a more optimal solution than the displacement fields within the Hill and Prandtl mechanisms, and also matches well with the published finite-element analysis results in this range. Beyond these loads, at greater footing settlements, or soil rigidity, the Prandtl mechanism is shown to be the most appropriate.

KEYWORDS: bearing capacity; clays; foundations; plasticity; settlement

INTRODUCTION

Centrifuge tests observing the behaviour of circular, shallow foundations on kaolin clay were performed in Cambridge (McMahon, 2012). The observed undrained displacement field was established using particle image velocimetry (PIV) (White *et al.*, 2003), and was found at typical working loads to resemble an ellipsoidal cavity-expansion mechanism more closely than the classical Prandtl (1921) or Hill (1949) mechanisms.

A theoretical, ellipsoidal cavity-expansion mechanism for perfectly rough, shallow, circular foundations was developed in McMahon *et al.* (2013). Figure 1(a) shows a comparison between the undrained deformation field observed in a centrifuge experiment and the ellipsoidal cavity-expansion mechanism. It can be seen that these mechanisms match well. McMahon *et al.* (2013) also present an energy method for a von Mises material with associated flow in three dimensions. Yield was determined using the von Mises yield criterion

$$(\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_2)^2 = 2\,q_u^2 \quad (1)$$

where q_u is the undrained strength in triaxial compression. This, along with an associated flow rule, implies that the incremental strain vector remains parallel to the deviatoric stress throughout the deformation process, providing a computationally simple evaluation of work. In order to facilitate comparison with previously published finite-element analyses, it was assumed that

$$q_u = 2c_u \quad (2)$$

Manuscript received 24 January 2013; revised manuscript accepted 19 July 2013. Published online ahead of print 10 September 2013.
Discussion on this paper closes on 1 May 2014, for further details see p. ii.
* Department of Engineering, University of Cambridge, Cambridge, UK.

This analysis procedure, together with the cavity-expansion mechanism, permitted the production of a theoretical upper-bound load–settlement curve for rough, circular, shallow foundations. The load–settlement curve for this mechanism does not reach a limiting value, and as such can be considered to lose its validity beyond normalised stress values (σ_f/c_u) similar to that of the classical bearing capacity factor of rough circular footings, $N_c = 6{\cdot}05$ (Eason & Shield, 1960).

The energy method developed by McMahon *et al.* (2013) can be validated against the results of published finite-element and finite-difference calculations on vertically loaded, circular, shallow foundations. These results will be compared with the analytical results later in the paper. A summary of the parameters used in the analyses is shown in Table 1.

McMahon *et al.* (2013) demonstrated that the ellipsoidal cavity-expansion mechanism gives a good fit to the finite-element data for values of σ_f/c_u between 4 and 6, but deviates significantly from the results outside this range. There are obviously other mechanisms that could be evaluated within this framework to determine an optimal upper bound. These will be evaluated in this paper.

Yadong *et al.* (2008) show displacement fields for a strip footing on sand at five different footing pressures. It can be seen from the data presented that the mechanism appears to transform from a cavity-expansion type mechanism at small loads to a mechanism at failure represented reasonably well by the Prandtl mechanism. While this research involved sands rather than the undrained clays discussed here, a similar trend of changing mechanisms at different load levels might well be observed.

ALTERNATE MECHANISMS

Levin (1955) developed a displacement field within the Hill mechanism in an upper-bound investigation of the ultimate indentation pressure of smooth, circular punches. This

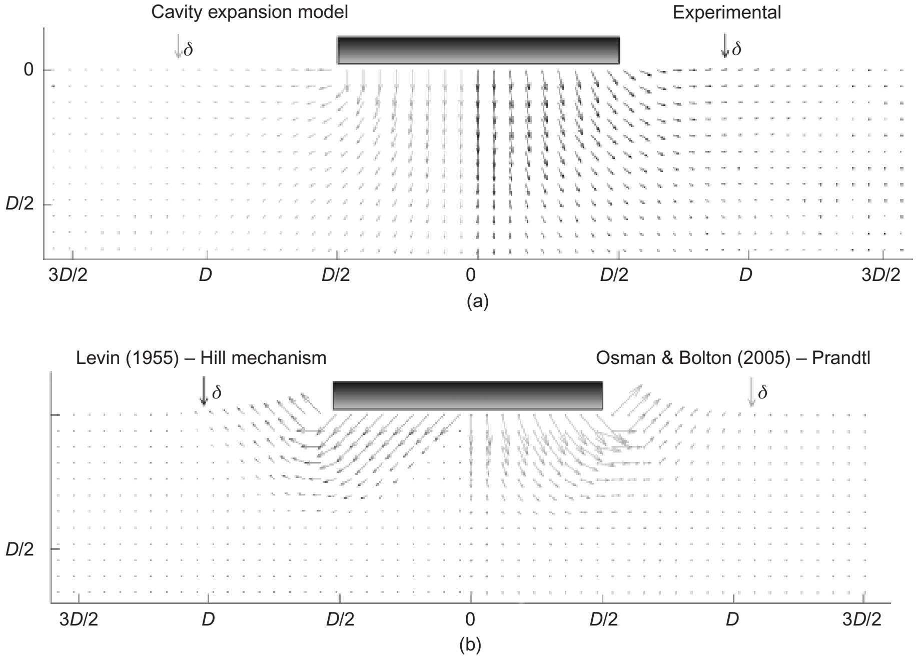

Fig. 1. Comparison of mechanisms: (a) centrifuge experiment and cavity-expansion model (McMahon *et al.*, 2013) mechanisms; (b) displacement fields of Levin (1955) within the Hill mechanism and Osman & Bolton (2005) within the Prandtl mechanism

Table 1. Summary of finite-element and finite-difference analysis results provided in the literature

Source	G/c_u	Footing roughness	Derived N_c
Taiebat & Carter (2000)	100	Rough	5·7
Taiebat & Carter (2010)	100	Rough	6·17
Gourvenec & Randolph (2002)	167	Rough	5·91
Klar & Osman (2008)	–	Smooth	5·63

displacement field is shown in Figure 1(b). As part of the development of mobilisable strength design (MSD), Osman & Bolton (2005) developed a kinematically admissible mechanism for a smooth, circular footing, which utilised the boundary of the classic Prandtl mechanism. By using the work equation of Shield & Drucker (1953) for an ideally plastic Tresca material, it was determined that $N_c = 5{\cdot}86$ for this mechanism. The primary difference between the Prandtl and the Hill mechanisms is the result of different active zones, which result in no movement below the centre of the Hill mechanism.

Klar & Osman (2008) further investigated the concept of MSD by using a process of energy minimisation for linear combinations of the elastic deformation field and those of either Hill or Prandtl in a process called extended MSD (EMSD). The Prandtl EMSD analysis was shown to be more optimal than that using the Hill mechanism.

Klar & Osman (2008) provide an expression for soil movements beneath a circular, shallow foundation in an elastic analysis based on integration of the classical solutions for elastic deformation beneath a point load. As this mechanism extends to infinity and involves an integral, making calculation computationally intensive, an approximation was generated by running a linear-elastic finite-element analysis of a vertically loaded, rough, rigid punch in Abaqus. The analysis used rigid boundaries at five foundation diameters in both the horizontal and vertical directions. The resulting deformation mechanism was exported for use in the energy approach.

The displacement fields within the Hill and Prandtl mechanisms, the cavity-expansion mechanism and the elastic mechanism are now used in the energy approach of McMahon *et al.* (2013) for soils of varying rigidity to evaluate an appropriate upper-bound at different levels of foundation loading. These are then evaluated against the finite-element analysis and finite-difference results of Taiebat & Carter (2000, 2010), Gourvenec & Randolph (2002) and Klar & Osman (2008).

RESULTS

A mesh size of $0{\cdot}2\%D$ was adopted for the analysis of the cavity-expansion, elastic, Hill and Prandtl mechanisms. As discussed in McMahon *et al.* (2013), the use of a finer mesh produced negligible changes and, therefore, did not justify the significant increase in computation time. A hemispherical radius of $2D$ was used for the cavity-expansion mechanism.

Analyses were performed on soils with G/c_u values between 10 and 10 000. It was found that the load–settlement behaviour fell on a single line when the normalised stress σ_f/c_u was plotted against the soil rigidity G/c_u multiplied by the normalised footing settlement δ/D for each mechanism. Figure 2 shows the results for the elastic, Hill, Prandtl and cavity-expansion mechanisms. The classical solution of Eason & Shield (1960) and the results from the finite-element analyses for rough footings of Taiebat & Carter (2000, 2010) and Gourvenec & Randolph (2002), and for a smooth footing from Klar & Osman (2008) are also shown. The elastic displacement field for smooth footings was also evaluated, but the results differed negligibly from those for the rough footing and so these results were omitted for clarity. It should be noted that the elastic mechanism line in

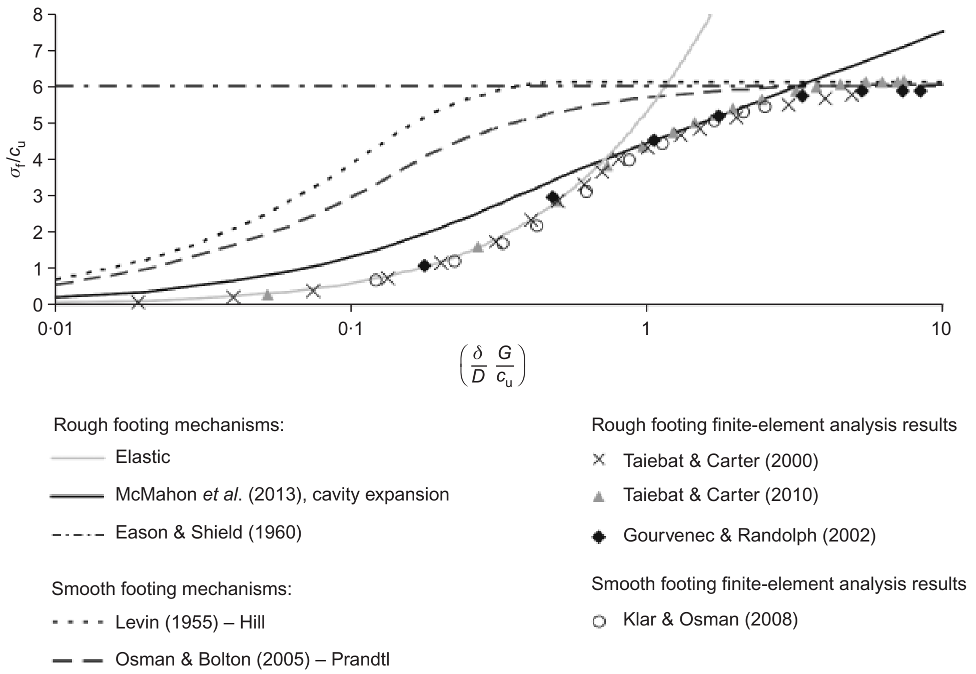

Fig. 2. Normalised footing stress (σ_f/c_u) plotted against ($\delta/D \times G/c_u$) for four mechanisms, the classical solution and published finite-element analysis results

Figure 2 is similar to the elastic settlement expression for a smooth footing of Davis & Selvadurai (1996)

$$\frac{\sigma_f}{c_u} = \frac{16}{\pi}\left(\frac{\delta}{D}\frac{G}{c_u}\right) \tag{3}$$

The results from the finite-element analyses show consistent results with only slightly different ultimate bearing capacity factor values. The elastic mechanism agrees very well with the finite-element analysis results up to $(\delta/D \times G/c_u) \approx 0{\cdot}7$, $\sigma_f/c_u \approx 4$. Beyond this point, up to a value of $(\delta/D \times G/c_u) \approx 3$, $\sigma_f/c_u \approx 6$, the cavity-expansion mechanism gives the lowest upper-bound solution and provides a good correlation with the published finite-element analysis results, in what could be considered the typical footing working range. Beyond this value the Prandtl mechanism is more appropriate, although it must be noted that this mechanism assumes a smooth foundation in contrast to the rough foundation finite-element results.

For all values of soil rigidity and settlement, the Prandtl mechanism provides a lower upper-bound solution than the Hill mechanism. The Prandtl mechanism converges to an ultimate bearing capacity of $N_c = 6{\cdot}11$, 4% greater than the value $N_c = 5{\cdot}86$ that was calculated in Osman & Bolton (2005). The Hill mechanism converges to an ultimate bearing capacity of $N_c = 6{\cdot}16$, approximately 5% greater than the value of $N_c = 5{\cdot}83$ presented by Klar & Osman (2008). These small differences are due to the von Mises yield criterion being used in the work presented here, rather than the Tresca criterion assumed by both Osman & Bolton (2005) and Klar & Osman (2008). In contrast to Tresca, the von Mises yield criterion allows work to be done by the intermediate strain component, resulting in a slight increase in work done in deforming the soil. The elastic and cavity expansion mechanisms do not give ultimate bearing capacities, showing continually increasing load with increasing deformation due to their infinite extent. In the upper-bound analyses presented here, the lowest upper bound at any settlement would be deemed to be the most appropriate mechanism for that settlement. At failure, the Prandtl mechanism becomes dominant, but for working loads, deformations might be more accurately reflected by the cavity-expansion mechanism.

A design line based on the lowest curve was produced and is shown in Figure 3, together with the results from the published finite-element analysis and finite-difference data. Also shown are the results of MSD and EMSD for the Prandtl mechanism from Klar & Osman (2008) and the elastic solution of Davis & Selvadurai (1996) (equation (3)). The discrepancy between the elastic solution and the new design curve for bearing stress $\sigma_f < 4c_u$ is small and probably the result of footing roughness. Figure 3 shows that the cavity-expansion mechanism matches well with the finite-element analysis data in the range $4c_u < \sigma_f < 6c_u$. The design curve, which embodies elastic analysis at small strains, elastic perfectly plastic cavity-expansion at intermediate strains, and ultimate bearing failure can, therefore, be used to describe the whole load–settlement behaviour of circular, shallow foundations on a homogeneous, linear-elastic perfectly plastic soil.

CONCLUSIONS

The energy method for a linear-elastic perfectly plastic soil employing the von Mises yield criterion with associated flow presented by McMahon *et al.* (2013) was used with an elastic mechanism and displacement fields within the classical Hill and Prandtl mechanisms to evaluate the optimal mechanism at different foundation loads. A parametric investigation was performed which demonstrated that, when the footing load was plotted against the rigidity index multiplied by normalised settlement, a single line was produced. The elastic mechanism provided excellent agreement with the published finite-element analysis in the small settlement region, while the cavity-expansion mechanism provided good agreement in the typical working region. The cavity-expansion mechanism, however, fails to reach an ultimate value, and beyond $(\delta/D \times G/c_u) \approx 3$, $\sigma_f/c_u \approx 6$ the Prandtl mechanism represents a better solution.

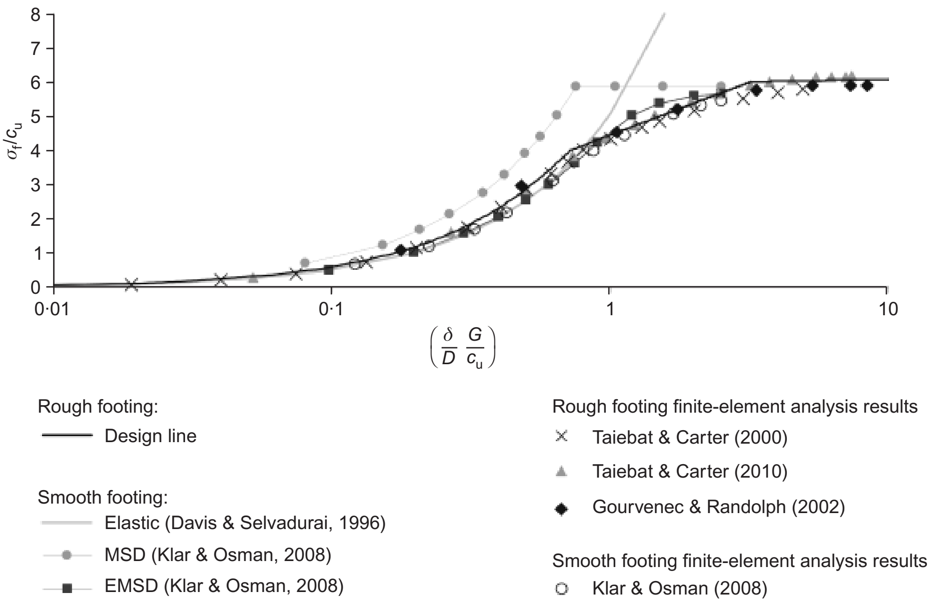

Fig. 3. Design line for rough, circular, shallow foundations in comparison to elastic, MSD and EMSD methods with published finite-element analysis results

ACKNOWLEDGEMENTS

The first author would like to acknowledge the financial support he received from the Cambridge Australia Trust (Poynton Scholarship) and the Principals of UK Universities (Overseas Research Students Awards Scheme) throughout his studies in Cambridge.

NOTATION

c_u undrained shear strength
D footing diameter
G shear modulus
M_c compatibility factor
N_c bearing capacity factor
q_u undrained strength in triaxial compression
δ footing settlement
σ_f footing pressure
σ_1, σ_2, σ_3 major, intermediate and minor principal stresses

REFERENCES

Davis, R. O. & Selvadurai, A. P. S. (1996). *Elasticity and geomechanics*. Cambridge, UK: Cambridge University Press.

Eason, G. & Shield, R. T. (1960). The plastic indentation of a semi-infinite solid by a perfectly rough circular punch. *ZAMP* **11**, No. 1, 33–43.

Gourvenec, S. & Randolph, M. (2002). Effect of strength non-homogeneity on the bearing capacity of circular skirted foundations subjected to combined loading. *Proceedings of the 11th international offshore polar engineering conference*, Kyushu, Japan, pp. 693–698.

Hill, R. (1949). The plastic yielding of notched bars under tension. *Q. J. Mech. Appl. Math*. **2**, No. 1, 40–52.

Klar, A. & Osman, A. S. (2008). Load–displacement solutions for piles and shallow foundations based on deformation fields and energy conservation. *Géotechnique* **58**, No. 7, 581–589, http://dx.doi.org/10.1680/geot.2007.00128.

Levin, E. (1955). Indentation pressure of a smooth circular punch. *Q. Appl. Math*. **13**, No. 2, 133–137.

McMahon, B. T. (2012). *Deformation mechanisms beneath shallow foundations*. PhD thesis, Cambridge University, Cambridge, UK.

McMahon, B. T., Haigh, S. K. & Bolton, M. D. (2013). Cavity expansion model for the bearing capacity and settlement of circular shallow foundations on clay. *Géotechnique* **63**, No. 9, 746–752, http://dx.doi.org/10.1680/geot.12.P.61.

Osman, A. S. & Bolton, M. D. (2005). Simple plasticity-based prediction of the undrained settlement of shallow circular foundations on clay. *Géotechnique* **55**, No. 6, 435–447, http://dx.doi.org/10.1680/geot.2005.55.6.435.

Prandtl, L. (1921). Über die Eindringungsfestigkeit Plastischer Baustoffe und die Festigkeit von Schneiden. *Z. Angew. Math. Mech*. **1**, No. 1, 15–20 (in German).

Shield, R. T. & Drucker, D. C. (1953). The application of limit analysis to punch-indentation problems. *J. Appl. Mech*. **20**, No. 4, 453–460.

Taiebat, H. A. & Carter, J. P. (2000). Numerical studies of the bearing capacity of shallow foundations on cohesive soil subjected to combined loading. *Géotechnique* **50**, No. 4, 409–418, http://dx.doi.org/10.1680/geot.2000.50.4.409.

Taiebat, H. A. & Carter, J. P. (2010). A failure surface for circular footings on cohesive soils. *Géotechnique* **60**, No. 4, 265–273, http://dx.doi.org/10.1680/geot.7.00062.

White, D. J., Take, W. A. & Bolton, M. D. (2003). Soil deformation measurement using particle image velocimetry (PIV) and photogrammetry. *Géotechnique* **53**, No. 7, 619–631, http://dx.doi.org/10.1680/geot.2003.53.7.619.

Yadong, C. H. E. N., Yuexina, S. H. E., Jiangdonga, C. A. I. & Jinminb, Z. A. I. (2008). Displacement field research of soil beneath shallow foundation based on digital image correlation method. *Proceedings of the SPIE international conference on experimental mechanics*, Nanjing, China, vol. 7375, paper 737550.