Practical methods to estimate the non-linear shear stiffness of fine grained soils

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ABSTRACT: The use of databases in geotechnical engineering allows engineers to make a priori estimates of soil behaviour. Based on a study of the published literature, a database of 20 clays and silts is presented that allows predictions to be made of the strain-dependent stiffness of fine-grained soils, based on simple soil parameters. The significance of rate effects is discussed and corrections are made. The use of a reference strain \( \gamma_{\text{ref}} \) to normalize shear strain values \( \gamma \) in relation to modulus reduction \( G/G_0 \) is discussed. Empirical formulations are presented based on a rigorous regression analysis, and design charts are constructed.

1. INTRODUCTION

In many applications of geotechnical engineering an estimate of the stiffness degradation of clays and silts is required. In earthquake engineering, prediction of the strain level that indicates modulus reduction is crucial in the prediction of damping. The seismic response is considered undrained in fine grained soils.

2. DATABASE

A database of clay and silt stiffness degradation was sourced from ten publications (listed in the Appendix). The data is presented in the terms of secant stiffness, the typical cyclic response being defined in Fig. 1. All the authors measured \( G_0 \) values directly apart from Teachavorasinskun et al (2002) who used the correlations in Hardin & Black (1968).

The samples derived from various countries and were tested in a variety of conditions from normally consolidated to heavily overconsolidated, in various laboratories and shear testing devices, over a period of 30 years. It should be recognised that most of this data relates to cyclic testing in which the immediately preceding strain history is one of reversal of the principal strain directions. The initial behaviour exhibited would therefore be expected to be one of maximum stiffness \( G_0 \) (Atkinson et al., 1990).

3. HYPERBOLIC MODELS

Konder (1963), Duncan & Chang (1970) and Hardin & Drnevich (1972) used hyperbolae to model shear stress-strain curves, being asymptotic to \( G_0 \) at zero strain and to \( \tau_{\text{max}} \) at infinite strain. By defining a reference strain \( \gamma_{\text{ref}} = \tau_{\text{ref}}/G_0 \) it was possible to rewrite the equation of a hyperbola as a normalised secant shear modulus \( (G/G_0) \) reducing with normalised shear strain \( (\gamma/\gamma_{\text{ref}}) \):
On the other hand, Fahey and Carter (1993) adopted the normalised shear strain ($\gamma/\gamma_{ref}$) to a power $\alpha$ in order to better fit the data of small strains: equation (3). This definition retains the feature that secant shear stiffness reduces to half its initial maximum value when $\gamma = \gamma_{ref}$. The current study will adopt the same family of modified hyperbolae in order to find an optimum fit for each soil.

$$\frac{G}{G_0} = \frac{1}{1 + (\frac{\gamma}{\gamma_{ref}})^{\alpha}}$$  \hspace{1cm} (3)

Darendeli (2001) presented a database of clay, sand and silt data. Using a Bayesian analysis, the curvature parameter $\alpha$ was found to be 0.92 for the normalised strain data. The reference strain was found to be a function of OCR, plasticity index and mean effective confining pressure. Zhang et al. (2005) similarly presented a database of sandy to clayey soils from South Carolina, North Carolina and Alabama. The curvature parameter $\alpha$ was shown to vary from about 0.6 to 1.55 for un-normalised shear strain data.

It must be recognised that the value of $\alpha$ will bear no relation to the strain rate used in the test. Fig. 2 shows equation (3) plotted with various values of the curvature parameter $\alpha$. It is observed that increasing $\alpha$ causes an increase in normalised stiffness at small normalised strains but decreases the stiffness at high strains. This behaviour is a feature of the modified hyperbolic model but it does not represent the typical behaviour of soil tested at different strain rates. Fine-grained soils typically show stiffness and strength enhancing at all strain rates, for strains in excess of the linear elastic limit: Vucetic & Tabata (2003).

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Table 1. Curve fitting parameters (original data)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$\gamma_{ref}$</th>
<th>$n$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>0.00540</td>
<td>73</td>
<td>1.69</td>
</tr>
<tr>
<td>min</td>
<td>0.00041</td>
<td>4</td>
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</tr>
<tr>
<td>mean</td>
<td>0.00138</td>
<td>18</td>
<td>0.97</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.00098</td>
<td>13</td>
<td>0.26</td>
</tr>
</tbody>
</table>

---

4.2 Rate effect corrections

It is has been known for many years that the stiffness and strength of clays is rate-sensitive. Richardson and Whitman (1963), for example, used triaxial tests with variable strain-rates. They demonstrated for normally consolidated plastic clay that an increased strain-rate led to enhanced stiffness at moderate strains without any change of pore pressure. In a recent review on the effect of strain rate on cyclic shear modulus at small strains (up to a shear strain of 0.01%) Vucetic & Tabata (2003) reported that the enhancement in stiffness per log$_{10}$ cycle of strain rate increased with plasticity index $I_p$ from about 2% for very low plasticity clays ($I_p < 10\%$) to about 5% for high
plasticity clays ($I_p = 40\%$). For large strains, however, Kulhawy & Mayne (1990) obtained a good correlation ($R^2 = 0.802$) for 26 clays by taking the undrained shear strength to increase by 10% per log$_{10}$ cycle of strain rate. Lo Presti et al (1997) and d’Onofrio et al (1999) offer evidence for low to medium plasticity clays ($10\% < I_p < 30\%$) which supports the proposition that the strain rate effect on stiffness may be negligible for very small strains, but can rise to about 5% per log$_{10}$ cycle at a strain of 0.01% and to about 10% per log$_{10}$ cycle at a strain of 1%. Detailed reviews of the influence of rate (viscous) effects at intermediate strain levels can be found, for example, in the keynote lectures of Tatsuoka & Shibuya (1992) and Tatsuoka et al (1997).

A carefully conducted undrained triaxial test achieving peak strength at an axial strain of about 2% (and therefore a shear strain of about 3%) after 8 hours would have a shear strain rate $\dot{\gamma} \approx 10^5$ s$^{-1}$. On the other hand, a resonant column vibrating under maximum excitation with a cyclic shear strain amplitude of 0.1% at 50 Hz would have a peak shear strain rate $\dot{\gamma} \approx 0.3$ s$^{-1}$, which is 5.5 log$_{10}$ cycles faster than the triaxial test.

The focus of this paper is stiffness at moderate strains. Accordingly, all stiffness data will be normalised to a standard test rate of $10^{-6}$ s$^{-1}$, by assuming a strain-rate effect of 5% per log$_{10}$ cycle consistent with the findings of Lo Presti et al (1997) and d’Onofrio et al (1999). In doing so it is accepted that the stiffness of very low plasticity clays at low cyclic strain amplitudes in resonant column tests is likely to be underestimated, and that the stiffness of high plasticity clays at large strain amplitudes in resonant column tests may remain overestimated. Nevertheless, the disparity in stiffness between dynamic and static test results should have been reduced.

Table 2 shows the transformed metrics for comparison with Table 1, rate-effects having been allowed for. The assumed test frequencies (unless given in the original publication) are given in the Appendix. Note the general reduction of the curvature parameter $\alpha$.

Figure 3 shows the original data compared with the rate-corrected data. The resonant column test curves are depressed to show a less-stiff response in the rate-corrected plot.

Table 2. Curve fitting parameters (rate corrected data)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$\gamma_{ref}$</th>
<th>$n$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>0.00336</td>
<td>73</td>
<td>1.01</td>
</tr>
<tr>
<td>min</td>
<td>0.00025</td>
<td>4</td>
<td>0.52</td>
</tr>
<tr>
<td>mean</td>
<td>0.00094</td>
<td>18</td>
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<tr>
<td>$\sigma$</td>
<td>0.00065</td>
<td>13</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Fig. 4 shows the hyperbolic fit to the normalised data once it has been rate corrected in the aforementioned manner. The following equation results:

$$\frac{G}{G_0} = \frac{1}{1 + \left( \frac{\dot{\gamma}}{\dot{\gamma}_{ref}} \right)^0.54}$$

$R^2=0.96$, $n=1105$, S.E. $= 0.0130$, $p<0.001$ (4)

The $R^2$ for this correlation is very good with 96 percent of the variation being explained by the model. The
Further empirical correlations must now be obtained for \( \gamma_{ref} \) in terms of readily available soil properties.

![Graph](image-url)  
Figure 4. \( \log_{10}(G_0/G^{-1}) \) versus \( \log_{10}(\gamma_{ref}/\gamma_0) \) rate corrected data (for key see Fig. 3)

4.3 Prediction of reference strain

Linear regressions were performed using the following variables: plasticity index, liquid limit and plastic limit and voids ratio. Fig. 5 shows the scatter-plots that display the data and the following regression equations:

\[
\gamma_{ref} = 2.17(I_p)/1000 \\
R^2 = 0.75, n = 61, S.E. = 0.00031, p < 0.001 \quad (5)
\]

\[
\gamma_{ref} = 1.25(w_L)/1000 \\
R^2 = 0.75, n = 61, S.E. = 0.00029, p < 0.001 \quad (6)
\]

\[
\gamma_{ref} = 2.73(w_P)/1000 \\
R^2 = 0.57, n = 61, S.E. = 0.00039, p < 0.001 \quad (7)
\]

\[
\gamma_{ref} = 0.56(e_0)/1000 \\
R^2 = 0.75, n = 61, S.E. = 0.00030, p < 0.001 \quad (8)
\]

Reasonable \( R^2 \) values are obtained for these correlations though an error band of ±50% (shown as dashed lines on Fig. 5) is commonly observed. In each case five of the London clay tests were deemed outliers to the trend. This may be due to the presence of fissuring in the samples: Gasparre (2005).

5. DESIGN CHARTS

5.1 Plasticity Index

Vucetic & Dobry (1991) presented commonly used design charts for seismic engineering. They emphasize the importance of plasticity index. A shortcoming of these charts is that they do not give a mathematical formulation for the degradation curves that they indicate.

5.2 Liquid Limit

The liquid limit (fall-cone) test is semi-automated and requires much less judgment on the part of the operator than is the case with the plastic limit test. A correlation with plasticity index \( (I_p = w_L - w_P) \) calls for both tests to be performed. The adoption of liquid limit alone as the parameter for new design charts should lead to greater reliability in practice. The Atterberg Limits \( w_L \) and \( w_P \) both relate to the capacity of clays to maintain an open stable structure with a high voids ratio. It is therefore no surprise that the correlations shown in Fig. 5 were found.

Voids ratio requires undisturbed samples in order to minimize water migration, but it offers no statistical improvement. Hence, \( w_L \) is favoured. Active clays have stronger intergranular attractions leading to the formation of well-bonded agglomerates. They accordingly tend to have high Atterberg Limits, and it is reasonable that they have been discovered to require higher strains to reduce their initial linear-elastic stiffness.

Fig. 7 shows new design curves for the degradation of clay and silt stiffness plotted against shear strain for a variety of liquid limits based on equations (4) and (6).

5.3 Accuracy of the model

Using equations (4) and (6), \( G/G_0 \) ratios were predicted for all the strain values in the database. Fig. 8 shows the plot of the predicted versus the measured data. Apart from the London clay outliers (reference strain values shown in Fig. 5) the modified hyperbola and the liquid limit predict \( G/G_0 \) within a bandwidth of ±30%, with lower accuracy at high strains. The framework presented in this paper is able to predict \( G/G_0 \) at any strain, for a clay or silt, to a reasonable degree of accuracy, with knowledge only of the liquid limit.

standard error (S.E.) is low and the probability of a correlation not existing (p) is less than 1 in 1000.
Figure 5. Clockwise from left - reference strain ($\gamma_{ref}$) versus plasticity index, liquid limit, plastic limit and voids ratio (for key see Figure 3)

Figure 6. Comparison of predictions using equations (4) & (5) with those from Vucetic & Dobry (1991)
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

G/G_0 predicted

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

G/G_0 measured

Figure 8. Accuracy of prediction model based on liquid limit (w_L) (for key see Fig. 3)

6. CONCLUSIONS

A detailed database of stiffness degradation has been analyzed to determine simple equations to estimate the behaviour of clays and silts for static, cyclic or dynamic applications. Liquid limit, plastic limit, plasticity index and voids ratio are shown to correlate well with reference strain (G/G_0).

Simply using the liquid limit and the derived modified hyperbola, the G/G_0 ratio of a clay or silt at any strain level can be predicted within ±30%. New design curves are drawn.

Depending on the engineering application and the site data available, engineers can make predictions of clay and silt stiffness degradation with confidence in the hyperbolic stress-strain curve and the relationships between reference strain and basic soil parameters.

The values of G/G_0 from equations (4) and (6), and in the design charts of Fig. 7, all refer to the normalized strain rate $\dot{\gamma}$ ≈ 10^6 s^{-1}. If values of G/G_0 were required for a different strain rate $\dot{\gamma}$, and for moderate strain amplitudes, then the scaling $G^* = G + 0.05(\log_{10}(10^6 \dot{\gamma}))$ could be used, following Vucetic & Tabata (2003).

7. ACKNOWLEDGEMENTS

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8. REFERENCES


### Appendix: Undrained Clay & Silt Database

<table>
<thead>
<tr>
<th>Reference</th>
<th>Test Type</th>
<th>Test frequency Assumed</th>
<th>Soils Studied</th>
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</thead>
<tbody>
<tr>
<td>Teachavorasinskun et al. (2002)</td>
<td>Cyclic Triaxial (CT)</td>
<td>given in paper</td>
<td>Bangkok Clay (3 sites)</td>
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<tr>
<td>Georgiannou et al. (1991)</td>
<td>RC, T &amp; TS</td>
<td>50Hz, 0.1Hz, 0.025Hz</td>
<td>Valericea Clay, Pietrafitta Clay, Todi Clay</td>
</tr>
<tr>
<td>Soga (1994)</td>
<td>Cyclic Triaxial (CT)</td>
<td>given in thesis</td>
<td>San Francisco Bay Mud, Pancone Clay</td>
</tr>
<tr>
<td>Shibuya &amp; Mitachi (1994)</td>
<td>Torsional Shear (TS)</td>
<td>given in paper</td>
<td>Hachiragata Clay</td>
</tr>
<tr>
<td>Rampello &amp; Silvestri (1993)</td>
<td>Resonant Column (RC)</td>
<td>50Hz</td>
<td>Valericea Clay, Pietrafitta Clay</td>
</tr>
<tr>
<td>Doroudian &amp; Vucetic (1999)</td>
<td>Direct Simple Shear (DSS)</td>
<td>0.025Hz</td>
<td>Santa Barbara Plastic Silt</td>
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<td>Gasparre (2005)</td>
<td>Triaxial (T)</td>
<td>given in thesis</td>
<td>London Clay II–Heathrow Terminal 5 project</td>
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