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The strength and dilatancy of sands

M. D. BOLTON (1986). *Géotechnique*, **36**, No. 1, 65–78

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An important factor has been neglected in deriving empirical relationships for various cohesionless soils between the peak angles of shearing resistance ϕ'_{max} and dilation ψ_{max} under plane strain conditions, i.e. the anisotropy in strength and deformation characteristics (Oda, 1981; Arthur & Assadi, 1977). Tatsuoka, Sakamoto, Kawamura & Fukushima (1986) have also performed a series of plane strain compression tests on air-pluviated samples of Toyoura sand (Tables 1 and 2). They also obtained a pronounced degree of anisotropy in both $\phi'_{max} = \arcsin[(\sigma_1' - \sigma_3')/(\sigma_1' + \sigma_3')]_{max}$ and $\psi_{max} = \arcsin[-(d\varepsilon_1 + d\varepsilon_3)/(d\varepsilon_1 - d\varepsilon_3)]_{max}$ at the peak stress condition (Figs 2 and 3). In these figures, sample densities are represented by e_0 which is the voids ratio value measured when the sample is at an isotropic pressure of 4·9 kN/m^2. A comparison of ϕ'_{max} and ψ_{max} at different pressure levels for the same value of e_0 is equivalent to a comparison between samples on a single isotropic compression line on an e–p' plane. The angle δ is defined in Fig. 4. Fig. 5 shows the relationship between ϕ'_{max} and ψ_{max} for all the data shown in Figs 2 and 3. It can be seen in Fig. 5 that the relationship for each value of δ is almost independent of σ_3'. However, these data clearly show that the relationship between ϕ'_{max} and ψ_{max} is affected by δ, whereas the effect of δ on this relationship is much less than the effects of δ on both the ϕ'_{max}–e_0 relationship and the ψ_{max}–e_0 relationship. Probably all the data which are used in the Paper are from plane strain compression tests at $\delta = 90°$ except the data from simple shear tests in which δ is around 40°. In the relationship between $\phi'_{max} - \phi'_{crit}$ and ψ_{max} shown in fig. 8 of the Paper, the simple shear data are located slightly below the average relationship for plane strain compression tests, probably at $\delta = 90°$. This departure is due partly to the difference in δ. Consequently, for anisotropic cohesionless soils, the relationship between ϕ'_{max}, ψ_{max} and the density index should involve a parameter expressing the anisotropy in strength and deformation characteristics.

Secondly, in the empirical relations proposed in the Paper, both ϕ'_{max} and ψ_{max} are a function of the logarithm of stress. It is shown in the Paper that Ottawa sand and Karlsruhe sand do not follow these relations. Fig. 6 shows the relationships between ϕ'_{max}, ψ_{max} and $\log \sigma_3'$ in plane strain at $\delta = 90°$ for Toyoura sand for $e_0 = 0{\cdot}7$ and $e_0 = 0{\cdot}8$ obtained from the data shown in Figs 2 and 3. However, the voids ratio change Δe for $e_0 = 0{\cdot}67$ during isotropic compression from $\sigma_c' = 5$ kN/m^2 to $\sigma_c' = 100$ kN/m^2 which is free from the errors due to membrane penetration and bedding errors is about $-0{\cdot}0030$ (Goto, 1984). This value was estimated by extrapolating the data from a series of isotropic compression tests on solid cylindrical samples of air-pluviated Toyoura sand of different sizes, i.e. height-to-diameter ratios of 63/30, 20/10, 15/7 and 10/5. It can be seen from Figs 2 and 3 that the changes in ϕ'_{max} and ψ_{max} from the change in e of $-0{\cdot}0030$ are of the order of 0·2–0·3° or less. Consequently, the shape of the relations shown in Fig. 6 is altered only very slightly when they are plotted for a constant consolidated voids ratio. It can be seen in Fig. 6 that in the low stress range where σ_3' is less than about 50 kN/m^2 (i.e. p' is less than about 130 kN/m^2) both ϕ'_{max} and ψ_{max} are almost independent of stress level. In Figs 2 and 3 this tendency can also be seen for the entire range of

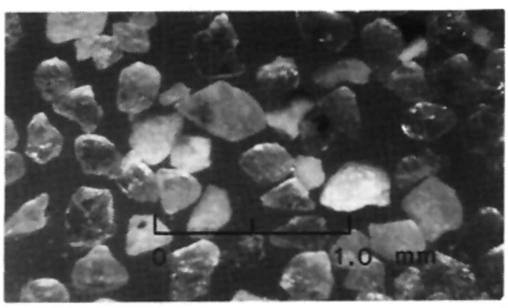

Fig. 1. Photomicrograph of Toyoura sand grains

Table 1. Index properties of Toyoura sand

Grain shape	Sub-angular*
Quartz†	≈90%
Chert	≈3%
G_s	2·645
D_{50}	≈0·16 mm
U_c	≈1·5
e_{max}‡	0·977
e_{min}‡	0·605

* Fig. 1.
† Yoshimi, Hatanaka & Oh-oka (1978).
‡ From the method specified by the Japanese Society of Soil Mechanics and Foundation Engineering.

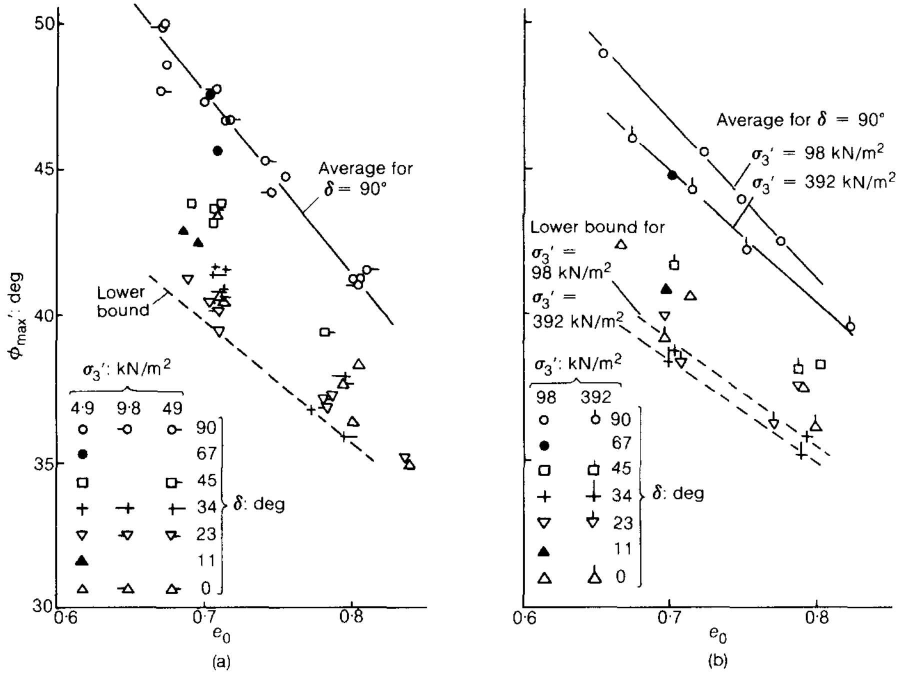

Fig. 2. Angle of shearing resistance ϕ'_{max} versus initial voids ratio e_0 measured at $\sigma_c' = 0{\cdot}05$ kgf/cm² (4·9 kN/m²) in plane strain for Toyoura sand: (a) σ_3' values of 4·9 kN/m², 9·8 kN/m² and 49·0 kN/m²; (b) σ_3' values of 98 kN/m² and 392 kN/m² (Tatsuoka, Sakamoto, Kawamura & Fukushima, 1986)

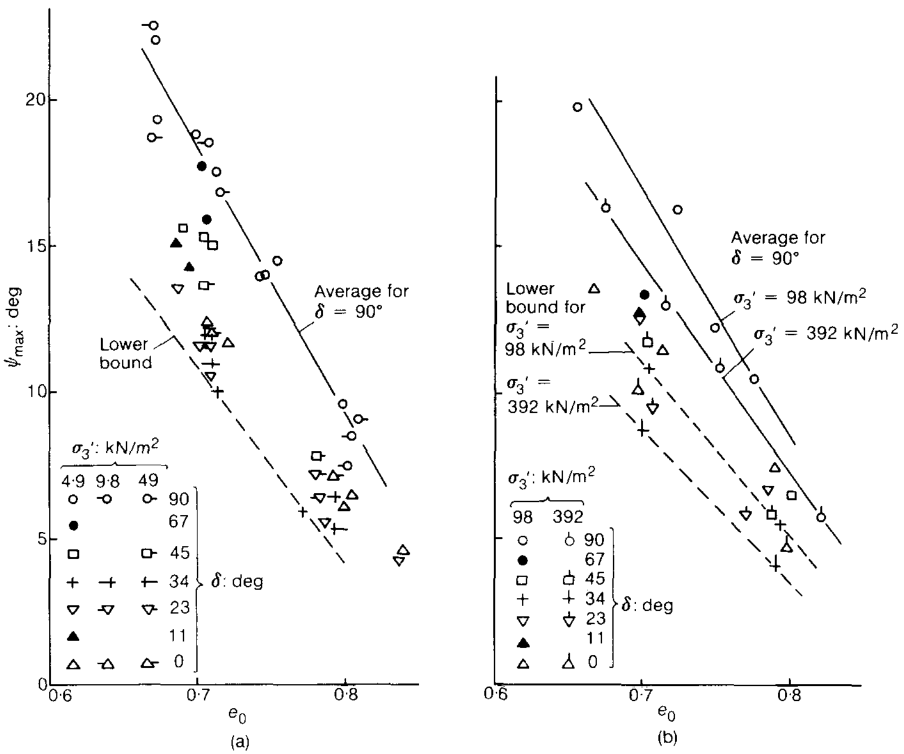

Fig. 3. Angle of dilatancy at peak stress condition ψ_{max} versus e_0 in plane strain for Toyoura sand: (a) σ_3' values of 4·9 kN/m², 9·8 kN/m² and 49·0 kN/m²; (b) σ_3' values of 98 kN/m² and 392 kN/m² (Tatsuoka, Sakamoto, Kawamura & Fukushima, 1986)

Table 2. Test conditions for Toyoura sand

Test condition	Sample dimensions: cm	End condition	Isotropic confining pressure: kN/m^2	ϕ'_{max}
Plane strain compression (Tatsuoka, Sakamoto Kawamura & Fukushima, 1986)	Height = 10·5, width = 4, length (σ'_2 direction) = 8	Well lubricated*	4·9, 9·8, 49, 98, 392	Corrected for membrane forces and others (method C-2-T)
Triaxial compression (Fukushima & Tatsuoka, 1984)	Height = 15, diameter = 7 (solid cylinder)	Well lubricated	2·0, 4·9, 9·8, 19·6, 49, 98, 196, 392†	Corrected for membrane forces defined at midheight of sample
Triaxial compression (Lam & Tatsuoka, 1986)	Height = 7·8, width = 7·8 (prismatic, square cross-section)	Well lubricated	98	Corrected for membrane forces

* Refer to Tatsuoka, Molenkamp, Torii & Hino (1984).
† σ_3' at failure was slightly different.

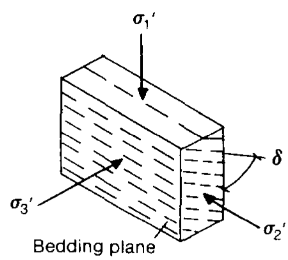

Fig. 4. Definition of angle δ

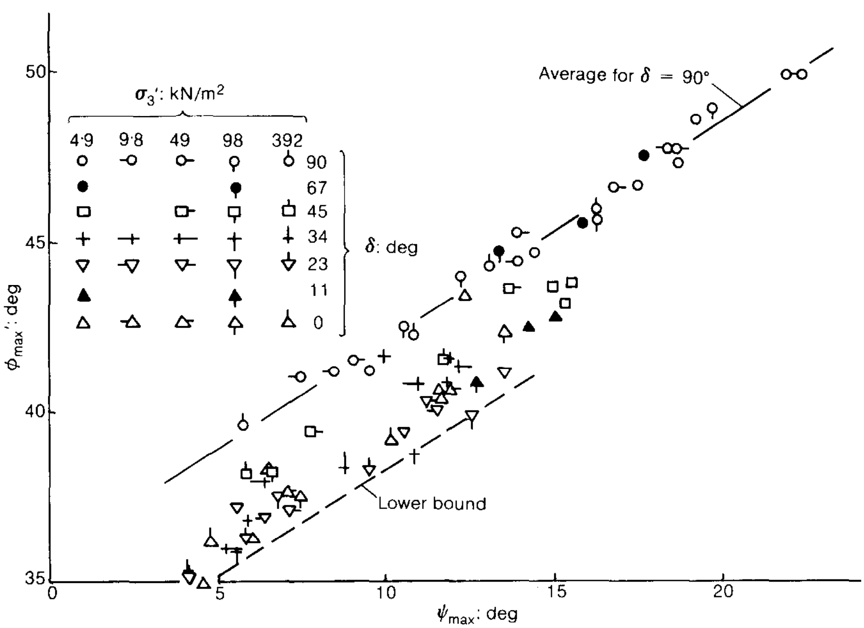

Fig. 5. Relationship between ϕ'_{max} and ψ_{max} in plane strain for Toyoura sand

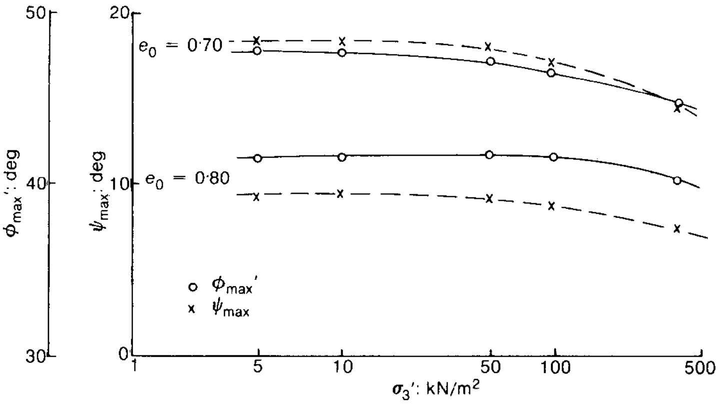

Fig. 6. Relationships between ϕ'_{max}, ψ_{max} and log σ_3' at $\delta = 90°$ in plane strain compression for Toyoura sand

densities and for the entire range of angles δ (i.e. 0–90°). A weak dependence of ϕ'_{max} and ψ_{max} on stress level at a stress range of $\sigma_3' \lesssim 50$ kN/m^2 for air-pluviated Toyoura sand has also been obtained in both triaxial compression tests (Fukushima & Tatsuoka, 1984) (also Fig. 7) and torsional shear tests (Tatsuoka, Sonoda, Hara, Fukushima & Pradhan, 1986). The dilatancy angle ψ_{max} shown in Figs 7(b) and 7(c) will be discussed later. The data for dense Leighton Buzzard sand at low stress levels shown in fig. 12 of the Paper also show the tendency that the rate of change in both ϕ'_{max} and ψ_{max} with ln p' becomes very small at very low stress levels. Therefore, as suggested in the Paper, it may be quite misleading without a basis of very reliable experimental results to assume that both ϕ'_{max} and ψ_{max} change in proportion to $-\ln p'$ in low stress ranges.

Finally, the data shown in fig. 7 of the Paper indicate that for the same density ϕ'_{max} is larger in plane strain than in triaxial compression, whereas for the same density $(d\varepsilon_v/d\varepsilon_1)_{max}$ is similar in both testing methods. However, the data for air-pluviated Toyoura sand, which were obtained at the Institute of Industrial Science, the University of Tokyo, show that the relationships for ϕ'_{max} and $(d\varepsilon_v/d\varepsilon_1)_{max}$ between plane strain and triaxial compression are not as simple as suggested in the Paper owing to the anisotropic mechanical properties of the sand. It can be seen in Fig. 8(a) that ϕ'_{max} is similar for plane strain and triaxial compression at an angle δ of about 30° where ϕ'_{max} becomes the minimum in plane strain.

In Figs 7(b), 7(c) and 8(b) the dilatancy angle ψ_{max} for triaxial compression is defined as

$$\psi_{max} = \arcsin\left[-\frac{(d\varepsilon_1/2 + d\varepsilon_3)}{(d\varepsilon_1/2 - d\varepsilon_3)}\right]_{max} \qquad (1)$$

The reason for this definition is as follows. It can easily be shown that a comparison of $d\varepsilon_v/d\varepsilon_1$ for triaxial compression and plane strain is equivalent to a comparison of ψ_{max} when ψ_{max} is given by equation (1) for triaxial compression and $\psi_{max} = \arcsin[-(d\varepsilon_1 + d\varepsilon_3)/(d\varepsilon_1 - d\varepsilon_3)]_{max}$ for plane strain. It is well known that the stress–dilatancy relations proposed by Rowe (1969)

$$\frac{\sigma_1}{\sigma_3} = K\left(-\frac{2\,d\varepsilon_3}{\varepsilon_1}\right) \qquad (2a)$$

for triaxial compression and

$$\frac{\sigma_1}{\sigma_3} = K\left(-\frac{d\varepsilon_3}{d\varepsilon_1}\right) \qquad (2b)$$

for plane strain fit most experimental data, whereas the value of K for plane strain is known to be slightly larger than that for triaxial compression. Thus it can easily be shown that a comparison of dilatancy characteristics between triaxial compression and plane strain in terms of K is equivalent to a comparison in terms of the ratio ϕ'_{max}/ψ_{max}. It can be seen in Fig. 8(b) that ψ_{max} is larger in plane strain than in triaxial compression at $\delta = 90°$. This point can also be seen by comparing Figs 3(a) and 3(b) with Fig. 7(b). However, the difference decreases to a minimum at $\delta \approx 30°$ as δ decreases as is the case for ϕ'_{max}. Therefore it is an oversimplification to assume that for the same density the value $(d\varepsilon_v/d\varepsilon_1)_{max}$ is similar for plane strain and triaxial compression. Furthermore, it can be seen by comparing Fig. 5 with Fig. 7(c) that to compare the value of K in equation (2) for triaxial compression and plane strain the anisotropy in strength and deformation characteristics should be taken into account.

In summary, the empirical relations for ϕ'_{max}

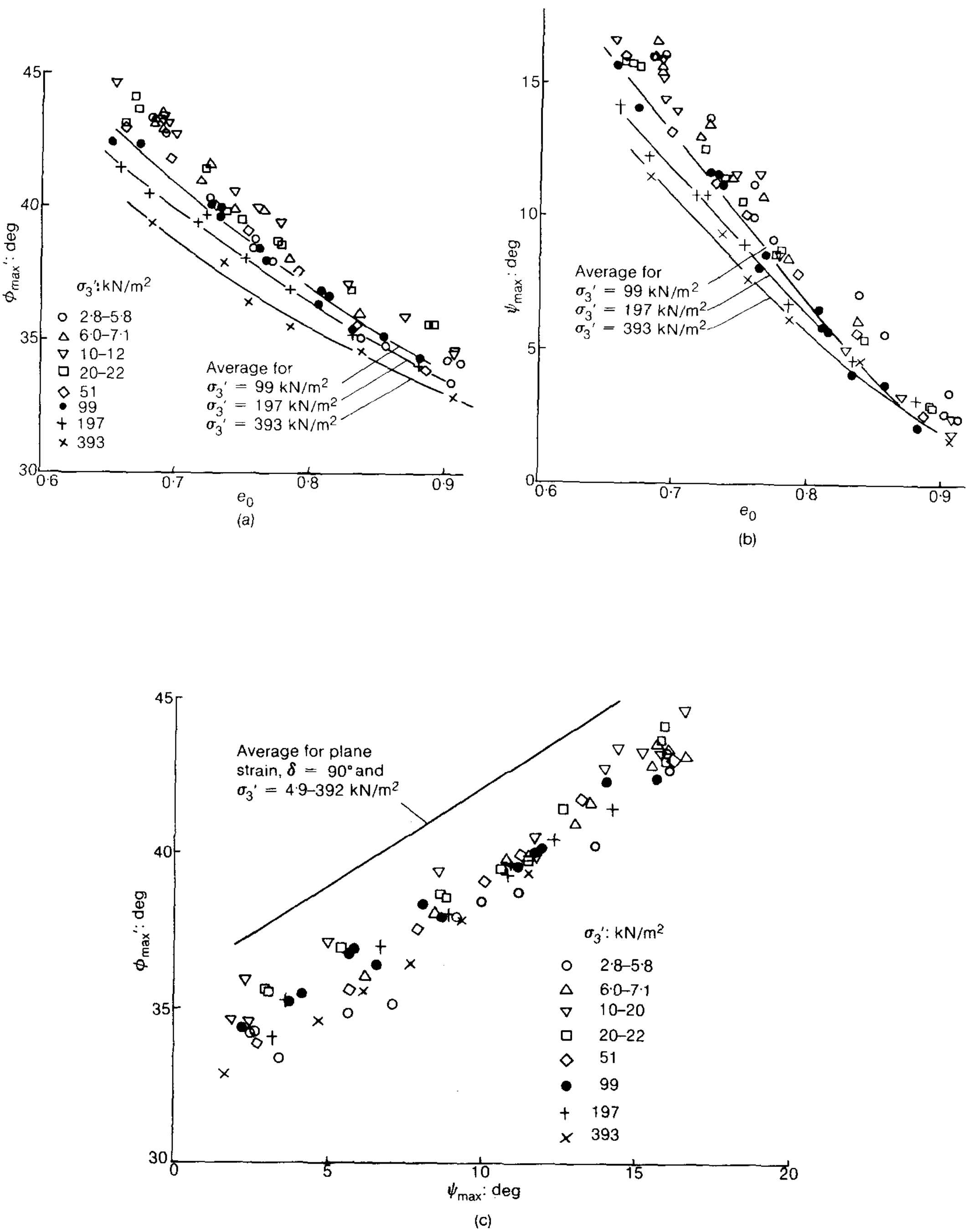

Fig. 7. (a) ϕ'_{max}; (b) ψ_{max}; (c) the relationship between ϕ'_{max} and ψ_{max} at $\delta = 90°$ in triaxial compression (Fukushima & Tatsuoka, 1984)

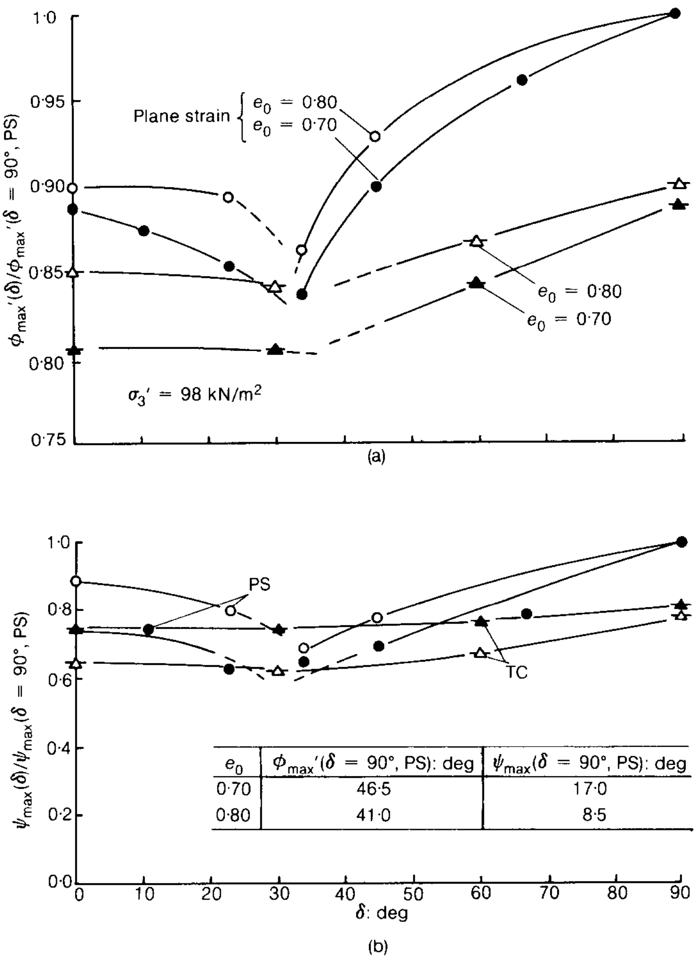

Fig. 8. ϕ'_{max} and ψ_{max} as a function of angle δ at $\sigma_3' = 98$ kN/m² in plane strain (PS) and triaxial compression (TC) (Lam & Tatsuoka, 1986)

and ψ_{max} proposed in the Paper can be put to more general use by taking into account both

(*a*) the anisotropy in ϕ'_{max} and ψ_{max}
(*b*) the low stress level dependence of ϕ'_{max} and ψ_{max}

at very low stress levels at least for such clean sands as Toyoura sand.

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Author's reply

The Writer has introduced an extensive and valuable body of recent data for Toyoura sand. For comparison with the empirical relations introduced in the Paper, it is necessary to select a value of ϕ_{crit}. This can, perhaps, be achieved by extrapolating Figs 5 and 7(c) of ϕ'_{max} against ψ_{max} to find ϕ_{max} at $\psi_{max} = 0$. A range of values can be determined from about 35° with the bedding at $\delta = 90°$ to 33° with $\delta = 30°$ in plane strain and about 32° in triaxial strain. In each case a scatter of $\pm 1°$ is discernible, which is typical.

If Rowe's stress dilatancy theory is fitted approximately, with lines of slope 0·8 following the Paper's fig. 6, it can then be seen that ϕ_{crit} is greater

(*a*) in plane strain than in triaxial strain
(*b*) at smaller confining pressure
(*c*) in looser packings
(*d*) when compressed perpendicularly to the bedding.

This accords with the view that strength is enhanced somewhat by local dilation even in a sample which is shearing at overall constant volume. Shear stresses will tend to be transmitted through the stronger particle assemblies which are locally dilating rather than the weaker assemblies which are contracting. This explanation also satisfies the observation of Norris (1977) that, for a sand of quartz grains at moderate stresses ($p' \approx 100$ kN/m^2), ϕ_{crit} increases with particle angularity from 29° when rounded to 40° when angular, the latter value reducing at higher stresses.

The 3° variation in ϕ_{crit} due to inherent anisotropy is rather larger than might hitherto have been supposed. This effect adds to the problem of selecting a value for ϕ_{crit}, but this uncertainty need not be overemphasized. A value of 34° is consistent with Norris's observation and fits all the Writer's data to $\pm 2°$.

Following this selection, the Paper's correlations fit the equivalent data of triaxial and plane strain compression perpendicular to the bedding ($\delta = 90°$) approximately as well as the average sand in the original survey. However, the Writer properly points out that his data at small confining pressures lead to the deduction that there is no appreciable further effect on soil behaviour when the mean stress p' is reduced below about 150 kN/m^2. The tendency to crush must be almost completely eliminated at these stress levels. It may be seen in Fig. 7(a), for example, that the further reduction of σ_3 from 50 kN/m^2 to 3 kN/m^2 (p' from about 100 kN/m^2 to 6 kN/m^2) does nothing to compensate for a low initial density. Indeed ϕ_{max} for $p' = 3$ kN/m^2 is about 1·5° smaller than that for $p' = 6$ kN/m^2, which is paradoxical. A prudent conclusion would be, in the absence of further verification for a particular sand, that the Paper's empirical correlation using I_R in equation (14) should be limited to the range of $p' > 150$ kN/m^2. Any smaller value of p' should be substituted by 150 kN/m^2. The original relative dilatancy index could then be rewritten

$$I_R = I_D\left[5 - \ln\left(\frac{p'}{150}\right)\right] - 1$$

for $p' > 150$ kN/m^2 and

$$I_R = 5I_D - 1$$

for $p' < 150$ kN/m^2. For dense soils ($I_D = 1$) this is an identical condition to the limit $I_R = 4$ suggested in the Paper: however, it is a stronger condition in the case of looser soils.

Perhaps the most striking aspect of the Writer's data is his demonstration of the effects of inherent anisotropy. Strength and dilatancy in plane strain reduce from a maximum for normal bedding, $\delta = 90°$, to a minimum when $\delta = 30°$. At this inclination of bedding, the plane strain data ($\delta = 30°$) had fallen as low as triaxial data for strength and dilatancy on normal samples ($\delta = 90°$). This interesting observation should be coupled with the deterioration in peak strength parameters following the induction of anisotropy due to principal stress reversal or rotation (Wong & Arthur, 1985). Further work on the strength anisotropy of other sands, with particles of varying sphericity and angularity, would be advantageous.

Clearly, the peak strength depends on the rate of dilatancy, which in turn depends on the geometry of particle movements in relation to the principal stress direction. The particle trajectories apparently depend on the mode of strain in three dimensions, the inherent anisotropy due to bedding and the anisotropy induced by strain history, even for samples at a given voids ratio confined under a given mean effective stress. Nor should the possible effects of progressive failure in the soil mass be forgotten when applying soil data to the design of geotechnical works.

The Writer has provided a valuable indication of the detailed behaviour of a sand at low stress levels, taking anisotropy into account—facets which were omitted from the Paper. Designers cannot rely uncritically on strengths in sands that are in excess of those measurable in conventional triaxial tests on normally bedded samples.

Indeed the uncertainties in density, bedding orientation, strain localization, stress distribution and strain path which will affect typical design decisions will often best be countered by a reliance solely on the critical state strength component, sacrificing any overall dilatancy which may or may not prove to be mobilizable.

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