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The strength and dilatancy of sands

M. D. BOLTON*

Extensive data of the strength and dilatancy of 17 sands in axisymmetric or plane strain at different densities and confining pressures are collated. The critical state angle of shearing resistance of soil which is shearing at constant volume is principally a function of mineralogy and can readily be determined experimentally within a margin of about 1°, being roughly 33° for quartz and 40° for feldspar. The extra angle of shearing of 'dense' soil is correlated to its rate of dilation and thence to its relative density and mean effective stress, combined in a new relative dilatancy index. The data of $\phi'_{max} - \phi'_{crit}$ in triaxial or plane strain are separately fitted within a typical margin of about 2°, though the strength of certain sands is underpredicted in the 1000–10 000 kN/m^2 range owing to the continued dilation of their crush-resistant grains. The practical consequences of these new correlations are assessed, with regard to both laboratory and field testing procedures.

L'auteur analyse de nombreuses données concernant la résistance et la dilatance de 17 sables sous déformation plane ou axisymétrique pour différentes densités et pressions d'étreinte. L'angle de résistance au cisaillement dans l'état critique d'un sol soumis au cisaillement à volume constant est principalement une fonction de la minéralogie et peut se déterminer facilement à 1° près, comme ayant une valeur d'environ 33° pour le quartz et 40° pour le feldspath. L'angle supplémentaire de cisaillement d'un sol dense dépend à sa vitesse de dilation donc de sa densité relative et de la contrainte effective moyenne, combinées dans un nouvel indice de dilatance relative. Les données de $\phi'_{max} - \phi'_{crit}$ en déformation plane ou triaxiale sont separées par une marge de 2° approximativement, bien que la résistance de certains sables soit sousestimée dans une fourchette de 1000–10 000 kN/m^2 en raison de la poursuite de la provoquée par l'écrasement de leurs grains résistants. L'article évalue les conséquences pratiques de ces nouvelles corrélations en ce qui concerne les méthodes d'essai en laboratoire et in-situ.

KEYWORDS: friction; sands; shear tests; soil properties; statistical analysis.

INTRODUCTION

Following the early work of D. W. Taylor, the strength and dilatancy of soils received a great deal of attention in the 1960s. Several schools of thought developed, the differences between which were more accentuated than the points in common. There has followed a growing agreement amongst those researching the strength of soil

(*a*) that secant, rather than tangent, ϕ' values should be the basis for discussion
(*b*) that dilatancy towards critical states is central to an understanding of soil behaviour
(*c*) that both effective stress and soil density affect the rate of dilatancy of soils and thereby their strength parameters.

That this understanding has failed to permeate more widely into practice can partly be blamed on the structure of the historic argument, which revolved around the theoretical relationship between strength and dilatancy. Since practitioners would usually be in the position either of measuring both or guessing both, this aspect of the dispute must have seemed sterile.

The failure to bridge the gap between research and practice has many serious consequences, however. Engineers often do not appreciate

(*a*) that a secant ϕ' value derived from a single triaxial test on a single specimen can offer a conservative parameter for design if the testing conditions are carefully chosen
(*b*) that the full range of soil strengths can be expressed in terms of the variation of (secant) ϕ' with density and stress
(*c*) that the conventional tangent parameters (c', ϕ') can only describe the full range of soil strengths if *both* are allowed to vary with density and stress
(*d*) that ignorance of the foregoing can lead to significant errors in predicting ultimate bearing stresses, for example.

The objectives of this Paper are as follows: firstly, to clarify the concepts of friction and dilatancy in relation to the selection of strength parameters for design; secondly, to introduce a new relative dilatancy index and to demonstrate its use in the prediction of the behaviour of sands at failure in relation to the available published data; thirdly to consider the implications of the new correlations to the procedures of laboratory and field testing.

Most of the concepts which are to be discussed can be applied with equal force to clays as to

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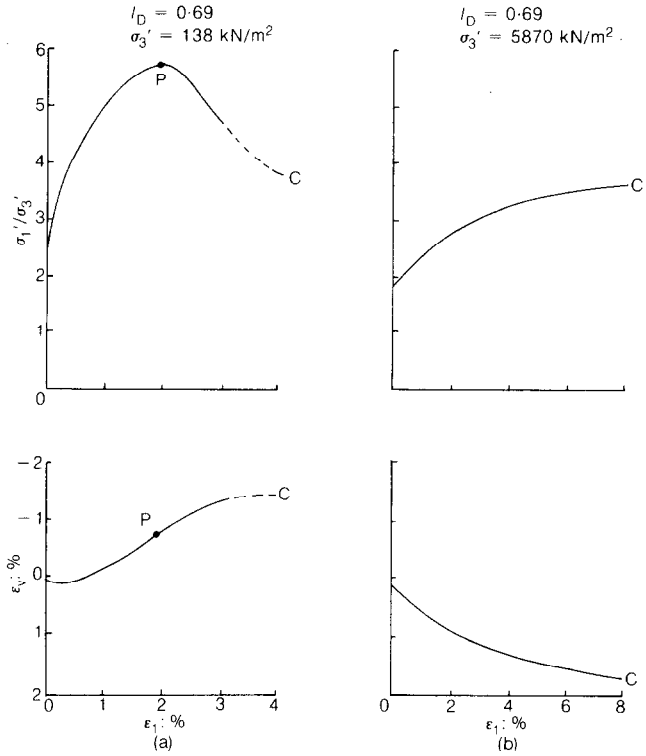

Fig. 1. Stress–strain behaviour of dense sand in plane compression (a) at low stress and (b) at high stress (Barden *et al.*, 1969)

sands. In particular, the explanation of 'true cohesion' as a by-product of dilatancy in overconsolidated clays is particularly fruitful (Rowe, Oates & Skermer, 1963). The present Paper is restricted in data, however, to those granular materials for which 'relative density' is a relevant and measurable parameter. Accordingly, the undrained strength problem has been set aside, and the discussion is focused on fully drained behaviour in terms of effective soil stresses.

ANGLES OF SHEARING AND DILATION

Data of a typical drained, plane strain, compression test (Barden, Ismail & Tong, 1969) on a dense, rectangular sample with frictionless ends is shown in Fig. 1(a). Strains were inferred from boundary displacements and volume changes, and they therefore underestimate the strains in the rupture zone which developed between points P and C. The achievement and accurate determination of the ultimate conditions at C are considerably hampered by the non-uniformity of the sample and the uncertainty regarding membrane correction following the formation of a rupture plane. Nevertheless such evidence as exists (Roscoe, 1970) suggests that soil in rupture zones will dilate fully to achieve a critical state, at which shear deformation can continue in the absence of a volume change.

The point of peak strength, P, is usually associated with the maximum rate of dilation defined as $(-\mathrm{d}\varepsilon_v/\mathrm{d}\varepsilon_1)_{max}$ where ε_v is the volumetric strain and ε_1 is the major principal strain (both defined positive in compression). The Mohr circles of effective stress, and differential strain, pertaining at peak P are shown in Fig. 2(a). The peak angles of shearing resistance ϕ'_{max} and dilation ψ_{max} are defined therein and given by

$$\sin \phi'_{max} = \left[\frac{\tau_{13}}{(\sigma_1' + \sigma_3')/2}\right]_p = \frac{(\sigma_1'/\sigma_3')_{max} - 1}{(\sigma_1'/\sigma_3')_{max} + 1} \quad (1)$$

$$\therefore \phi'_{max} = 44{\cdot}8^\circ$$

$$\sin \psi_{max} = \left(-\frac{\mathrm{d}\varepsilon_v}{\mathrm{d}\gamma_{13}}\right)_p = -\frac{(\mathrm{d}\varepsilon_1/d\varepsilon_3)_{max} + 1}{(\mathrm{d}\varepsilon_1/\mathrm{d}\varepsilon_3)_{max} - 1} \quad (2)$$

$$\therefore \psi_{max} = 14{\cdot}7^\circ$$

It should particularly be noted that ϕ' herein refers to the 'secant' angle of shearing obtained by dropping a tangent from the origin on to a single Mohr circle of effective stress.

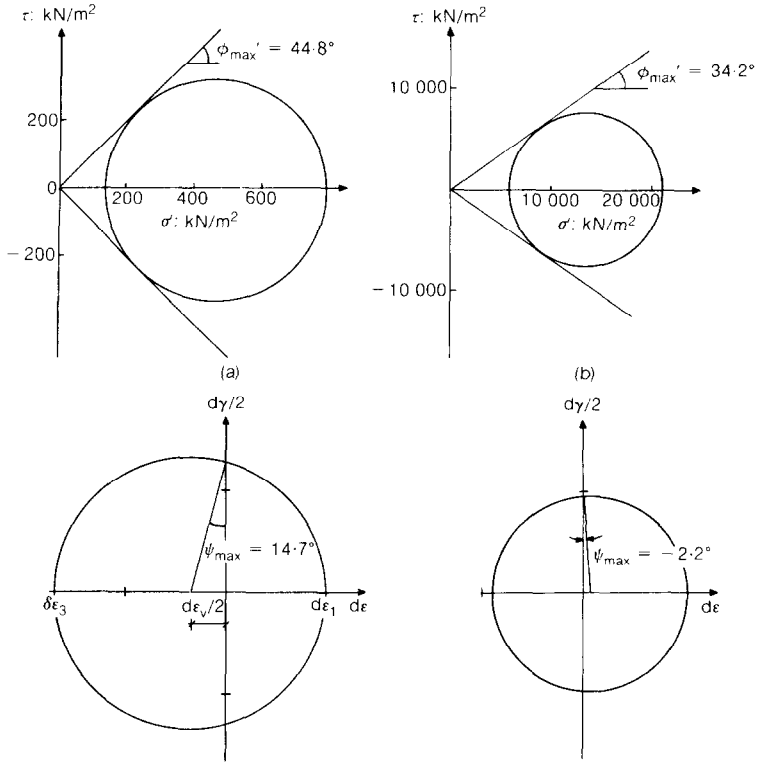

Fig. 2. Mohr circles of stress and strain increments of dense sand at maximum stress ratio in Fig. 1 (a) at low stress and (b) at high stress

A similar plane strain test on dense sand under extreme stresses mimics the normal behaviour of very loose sand and appears in Fig. 1(b), with the data for point C at $\varepsilon_1 = 8\%$ presented in Mohr circles in Fig. 2(b). The peak phenomenon is effectively suppressed while the sample volume contracts rather than dilates. The end state of the whole sample at C is analogous to that which can be observed in the thin rupture zones of the dilatant sample, tending towards a critical state

$$\phi'_{\text{crit}} = 35^\circ$$

$$\psi_{\text{crit}} = 0$$

The mechanical significance of the angle of dilation in a plane strain deformation can best be appreciated by assuming that the Mohr circle of plane strain increments in Fig. 2(a) can be applied to the case of direct shear shown in Fig. 3. If rigid blocks of non-failing soil are assumed to bound the thin uniformly straining rupture zone ZZ, this must mean that for compatibility ZZ must be a zero extension line so that

$$d\varepsilon_z = 0 \qquad (3)$$

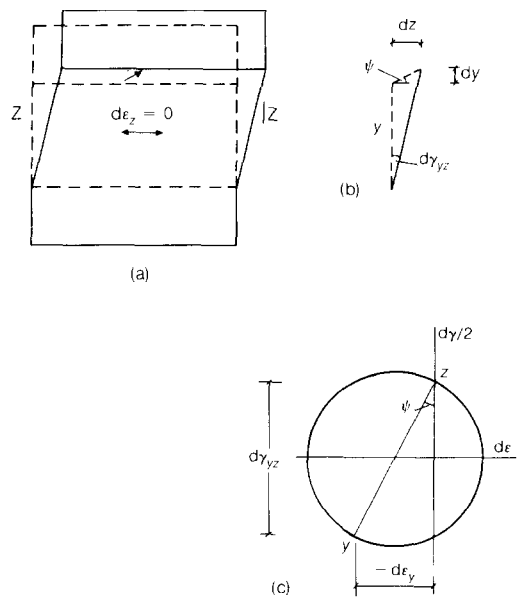

Fig. 3. Angle of dilation ψ in plane shear

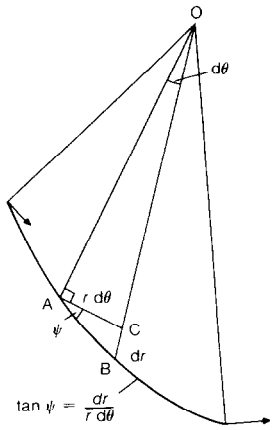

Fig. 4. Logarithmic spiral slip surface

within the rupture zone. Also

$$d\gamma_{yz} = \frac{dz}{y}$$

$$d\varepsilon_y = -\frac{dy}{y}$$

so in Fig. 3(b)

$$\tan\psi = -\frac{d\varepsilon_y}{d\gamma_{yz}} = \frac{dy}{dz} \tag{4}$$

The angle of dilation is then seen in Fig. 3(b) to be equal to the instantaneous angle of motion of the sliding blocks relative to the rupture surface.

Figure 4 demonstrates that the consequence of the assumption of a constant angle of dilation on a slip surface is the replacement of slip circles by the logarithmic spiral slip surface between rigid zones, since for a small angle AOB = $d\theta$

$$CAB = \psi$$

$$CB = dr$$

$$AC = r\,d\theta$$

$$\therefore \tan\psi = \frac{dr}{r\,d\theta}$$

so that

$$\frac{dr}{r} = \tan\psi\,d\theta \tag{5}$$

On integrating

$$r = r_0 \exp(\theta \tan\psi) \tag{6}$$

where $r = r_0$ at $\theta = 0$. At $\psi = 0$ the well-known slip circle $r = r_0$ is indicated. The larger is ψ the larger is the 'downstream' radius of the rupture line for a given 'upstream' excitation at $r = r_0$. If a relative rotation of the two supposedly rigid zones then occurs during collapse, the ensuing downstream displacements are simply proportional to the radius and are therefore correspondingly increased as ψ increases. In practice, it is known that failures can occur progressively, with different mobilizations of strength and dilatancy at various locations along a developing slip surface.

If rupture figures comprising a mosaic of sliding wedges with plane faces are alternatively assumed, the effect on the stability calculations of increasing ψ at constant ϕ' is easily shown *either* to be zero if the mosaic is still capable of sliding in the manner intended *or* to cause a seizure of the mosaic due to enhanced wedging which can make a previously critical slip mechanism kinematically inadmissible. Depending on the particular geometry of the problem therefore, the effect of increasing ψ at constant ϕ' may either be neutral or beneficial to the overall stability. A safe strategy must therefore be to err, if at all, by underestimating both ϕ' and ψ in any subsequent analysis.

A great deal of attention has been focused on the relation between ϕ' and ψ, and in particular between ϕ'_{max} and ψ_{max}. The stress–dilatancy theory of Rowe (De Josselin de Jong, 1976; Rowe, 1962, 1969) has proved to possess great explanatory power, as well as a close approach to the published data. Since the intention here is principally to introduce and correlate data, it is unnecessary to expand in detail on the theory. It will be valuable, however, to employ a less rigorous but similar approach to develop an expression which deviates only slightly from Rowe's in its estimate of ϕ' appropriate to a plane shear test.

Suppose that ϕ'_{crit} is the angle of shearing observed in a simple shear test on soil loose enough to be in a critical state, with zero dilation. Now suppose, following Fig. 5, that the same soil is tested dense, so that overriding at points of contact must occur unless the particles crush. Suppose that the particles above the overall zero-extension line ZZ form one rigid zone sliding upwards at ψ over the rigid zone beneath, in accordance with the external observation of a dilatancy angle ψ. Assume that the angle of shearing developed on the inclined microfacets SS, on which there is zero dilation, remains at ϕ'_{crit}. Since all the sliding now takes place on surfaces parallel to SS it is permissible to view the observed angle of shearing ϕ' on the rupture surface as comprising the two components ϕ'_{crit} and ψ as shown in Fig. 5. This specifies

$$\phi' = \phi'_{crit} + \psi \tag{7}$$

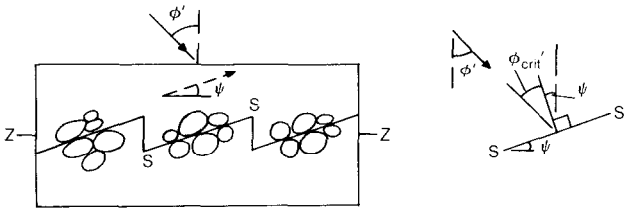

Fig. 5. The saw blades model of dilatancy

as an elementary friction–dilatancy relation in which no attempt has been made either to optimize the failure mechanism or to correlate the directions of principal stress and strain increments.

Figure 6 compares equation (7) with Rowe's stress–dilatancy relation for plane strain

$$\frac{\sigma_1'}{\sigma_3'} = \left(\frac{\sigma_1'}{\sigma_3'}\right)_{\text{crit}}\left(1 - \frac{d\varepsilon_v}{d\varepsilon_1}\right) \tag{8}$$

in the particular case $\phi'_{\text{crit}} = 33°$ corresponding to the typical value for quartz sands, and using equations (1) and (2) for ϕ' and ψ respectively. It will be seen that equation (7) overestimates $\phi' - \phi'_{\text{crit}}$ compared with equation (8) by about 20%. In other words, Rowe's stress–dilatancy relationship for plane shear, over the range of ψ shown in Fig. 6, is operationally indistinguishable from

$$\phi' = \phi'_{\text{crit}} + 0{\cdot}8\psi \tag{9}$$

The advantage of having developed expressions such as (7) and (9) is that any angle of shearing in excess of the friction angle of loose earth is seen to be due solely to the geometry of the volumetric expansion which is necessary before shearing can take place.

EFFECTS OF DENSITY AND CONFINING STRESS

The accepted definition for the state of compaction of granular materials is relative density

$$I_D = \frac{e_{\max} - e}{e_{\max} - e_{\min}} \tag{10}$$

where $e_{\max}$ is defined as the voids ratio achieved in quickly inverting a measuring cylinder containing the dry soil and $e_{\min}$ is that achieved under optimal vibration of a compactive mass under saturated conditions and without causing crushing. It has generally been found (Cornforth, 1973) that relative density offers a superior correlation compared with voids ratio for the strength of sands, presumably since it compensates for effects of particle grading and shape which influence $e_{\max}$ and $e_{\min}$.

If soil particles were perfectly strong and rigid, the tendency towards dilation would be solely a function of the density and arrangement of the particle structure. However, experiments with steel shot, which were effectively unbreakable but were deformable plastically, have shown (Bishop, 1972) that increased confining pressure leads to reduced angle of shearing. It has also been demonstrated (Bishop, 1972; Billam, 1972; Vesic & Clough, 1968) by tests on granular soils at elevated pressures that particle crushing occurs, thereby reducing the observed maximum angles of dilation and shearing, $\psi_{\max}$ and $\phi'_{\max}$, for a given initial density. Soil particles may crush before they override.

No completely consistent treatment of both density and confining pressure has previously been undertaken in producing empirical relations by which $\phi'_{\max}$ could be predicted. This is now attempted. To achieve a given angle of dilatancy, it is argued, the particle structure should *both* be dense *and* not so highly stressed that asperities should fracture in preference to overriding. This suggests a relation in which a density parameter is multiplied by a stress parameter. Since the logarithm of stress has been shown (Billam, 1972; Vesic & Clough, 1968) to effect a linear reduction $\phi'_{\max}$, an expression of the form

$$\psi_{\max} = AI_D \ln\left(\frac{p'_{\text{crit}}}{p'}\right) \tag{11}$$

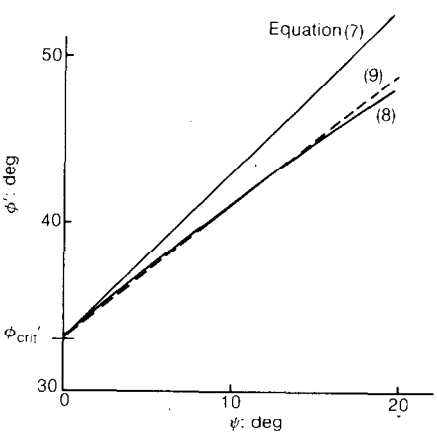

Fig. 6. Stress–dilatancy relations

Table 1. Sand data

Identification	Name	d_{60}: mm	d_{10}: mm	e_{min}	e_{max}	ϕ'_{crit}	Reference
A	Brasted river	0·29	0·12	0·47	0·79	32·6	Cornforth (1964, 1973)
B	Limassol marine	0·11	0·003	0·57	1·18	34·4	Cornforth (1973)
C	Mersey river	≈0·2	≈0·1	0·49	0·82	32·0	Rowe (1969) Rowe & Barden (1964)
D	Monterey no. 20	≈0·3	≈0·15	0·57	0·78	36·9	Marachi, Chan, Seed & Duncan (1969)
E	Monterey no. 0	≈0·5	≈0·3	0·57	0·86	37·0	Lade & Duncan (1973)
F	Ham river	0·25	0·16	0·59	0·92	33·0	Bishop & Green (1965)
G	Leighton Buzzard 14/25	0·85	0·65	0·49	0·79	35·0	Stroud (1971)
H	Welland river	0·14	0·10	0·62	0·94	35·0	Barden *et al.* (1969)
I	Chattahoochee river	0·47	0·21	0·61	1·10	32·5	Vesic & Clough (1968)
J	Mol	0·21	0·14	0·56	0·89	32·5	Ladanyi (1960)
K	Berlin	0·25	0·11	0·46	0·75	33·0	De Beer (1965)
L	Guinea marine	0·41	0·16	0·52	0·90	33·0	Cornforth (1973)
M	Portland river	0·36	0·23	0·63	1·10	36·1	Cornforth (1973)
N	Glacial outwash sand	0·9	0·15	0·41	0·84	37·0	Hirschfield & Poulos (1964)
P	Karlsruhe medium sand	0·38	0·20	0·54	0·82	34·0	Hettler (1981)
R	Sacramento river	0·22	0·15	0·61	1·03	33·3	Lee & Seed (1967)
S	Ottawa sand	0·76	0·65	0·49	≈0·8	30·0	Lee & Seed (1967)

is implied, in which p' is the mean effective stress at failure, p'_{crit} is the elevated stress just sufficient to eliminate dilation by crushing and A is a constant. However, the quantity of data for ϕ'_{max} is rather greater than that for ψ_{max}, and that for triaxial tests greatly exceeded that for plane strain tests, and therefore an approach must be used which can embrace these additional data.

It should be recognized that the angle of dilatancy becomes a meaningless parameter in an axisymmetric triaxial compression test, since the geometrical relationship of Fig. 2 applies only to plane strain. Indeed a Mohr circle of strain increments for the densest sand in triaxial compression would indicate that mean strains in any vertical plane were contractile rather than dilatant, owing to the implied distribution of lateral strain amongst all the available directions. It is therefore necessary to revert to $(-d\varepsilon_v/d\varepsilon_1)_{max}$ as the useful measure of triaxial dilatancy rate.

Table 1 assembles the characteristics of the sands used in the correlation study, comprising a large proportion of the relevant data available in the literature and emanating from well-established laboratories. The data refer to samples tested in compression with an initial height-to-width ratio of 2, unless indicated otherwise. Some researchers failed to disclose the precise conditions of the test platens, and it must be assumed that some of the variations in results may be accounted for by variations in end friction. The wide range of grading classifications is made evident by the d_{60} and d_{10} sizes which have been quoted; estimates have been made in certain cases where these particular values were not given by the researchers. Figures for e_{max} and e_{min} were essential to the study. Tavenas & La Rochelle (1972) have shown that variations in limiting voids ratios due to variations in technique between the American Society for Testing and Materials' methods and those of Kolbuszewski should not usually exceed 0·02. This would imply an experimental error band for relative density determinations of the order of $\pm 0{\cdot}05$.

The critical state angle of shearing was central to a rational portrayal of the data. The ideal method of determination for ϕ'_{crit} is from extrapolation of a series of values of ϕ'_{max} recorded in compression tests on soil samples of various densities, in which the rate of dilation at failure was observed, so that a value consistent with zero dilation at failure could be determined. In each case quoted in Table 1, it was felt that sufficient information was available to make a judgement on ϕ'_{crit} which should not have been in error by more than 1°. The observed range for the sands was from 32·0° to 37·0°, the higher values always pertaining to sands which were said to contain a significant proportion of feldspar, the lower values pertaining to quartz sands. This was consistent with previous findings (Koerner, 1970; Lee, 1966) that ϕ'_{crit} for felspathic sands was of the order of 40°.

An initial perusal of the test data showed that equation (11) was insufficient as a measure of dilatancy potential, since p'_{crit} appeared to reduce with reducing relative density, and at an increasing rate at very small relative densities. A relative dilatancy index of the form

$$I_R = I_D(Q - \ln p') - R \qquad (12)$$

was therefore studied, since this function returns a zero value at some pressure p'_{crit} such that

$$\ln p'_{crit} = Q - \frac{R}{I_D} \tag{13}$$

(Here, the value for Q will depend on the units taken for p': kilonewtons per metre squared will be used here). This expression has the advantage of ensuring that zero dilatancy is achieved at a critical effective stress which itself reduces strongly when the relative density I_D takes small values. Beyond these considerations, however, any preference for a particular definition of a relative dilatancy index must be entirely empirical.

It was found that values $Q = 10$ and $R = 1$ created a definition for a relative dilatancy index

$$I_R = I_D(10 - \ln p') - 1 \tag{14}$$

which apparently offered a unique set of correlations for the dilatancy-related behaviour of each of the sands in laboratory element tests. The following correlations were found to be available in the range $0 < I_R < 4$. For plane strain

$$\phi'_{max} - \phi'_{crit} = 0{\cdot}8\psi_{max} = 5I_R{}^{\circ} \tag{15}$$

For triaxial strain

$$\phi'_{max} - \phi'_{crit} = 3I_R{}^{\circ} \tag{16}$$

For both test configurations

$$\left(-\frac{d\varepsilon_v}{d\varepsilon_1}\right)_{max} = 0{\cdot}3I_R \tag{17}$$

It should be recognized that equations (15) and (17) permit alternative predictions of the rate of dilatancy in plane strain, although manipulation of equation (2) would reveal that the difference is numerically insignificant in the range of interest.

The justification for these relations now follows, consisting of the test data of the sands listed in Table 1 presented to display

(*a*) the variation in the peak angle of shearing resistance, and the rate of dilatancy, with relative density at a particular stress level
(*b*) the variation in peak angle of shearing resistance with stress level, at particular relative densities
(*c*) the combined variation for particular sands in those few studies where both density and stress level effects were studied
(*d*) the comparison in behaviour between plane strain and triaxial test behaviour.

Figure 7 displays the available data of both $\phi'_{max} - \phi'_{crit}$ and dilatancy rate $(-d\varepsilon_v/d\varepsilon_1)_{max}$ versus initial relative density, at what is intended to be a constant mean effective stress p' at failure

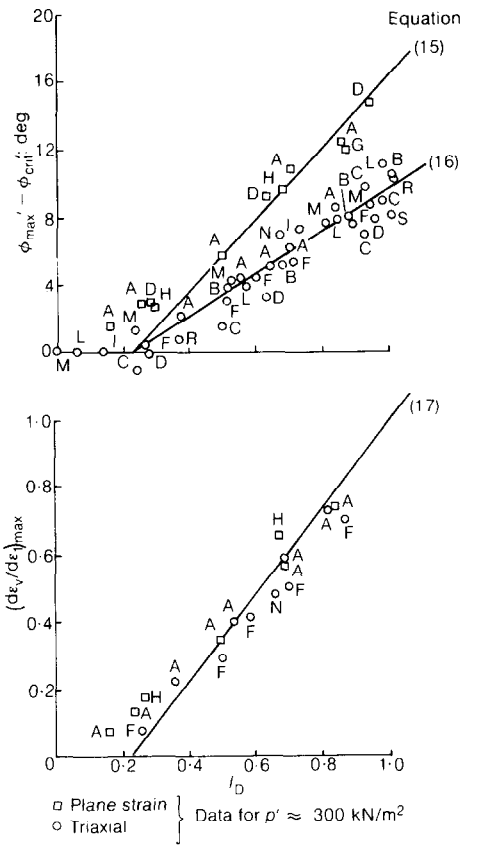

Fig. 7. Data for sands in Table 1 at failure with p' in the range 150–600 kN/m²

of 300 kN/m². To trap a sufficient variety of data it was necessary to accept data falling within a factor of 2 of this stress: this has been plotted without modification. However, some data from the longer test series have been omitted for parity, where they simply confirm the trend and scatter of results already plotted from the same series.

The pattern of plane strain strengths lying above triaxial strengths has long been recognized and was noted by many of the researchers listed in Table 1. The apparently linear correlation of particular sands on these axes has also been noted (Bishop, 1972), but the superimposition of data of such a variety of sands taking advantage of ϕ'_{crit} as a baseline was encouraging. The uniqueness of data of dilatancy rate irrespective of test mode has also been widely reported (Cornforth, 1973).

The empirical relations (14)–(17) appear as converging straight lines in Fig. 7, offering the concept of zero dilation in soils with an initial relative density of 0·23 at this effective stress level.

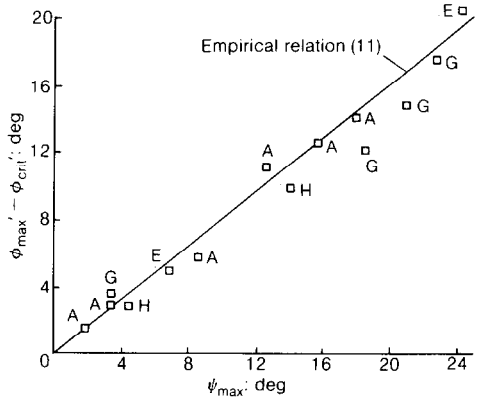

Fig. 8. Angles of shearing resistance and dilatancy in plane strain

It will be seen that the lines are not apparently of optimal fit since particular weightings have been attached to the data. Where a sequence of compression tests has been carried out at various densities, for example on soil A, it is usual for the experimenter to adopt a constant value of the cell pressure providing σ_3'. This leads to enhanced values of the mean effective stress at higher relative densities, due to the attendant increase in σ_1'/σ_3'. It is consistent with the later data of stress dependence to suppose that data of $\phi'_{max} - \phi'_{crit}$ in the vicinity of $I_D = 1$ could be relatively reduced by 2° on this account, with a corresponding shortfall in dilatancy rate.

Furthermore, the empirical relations have been selected taking into account the whole spread of data depicted in Figs 7–11, the required linkages between parameters suggested by stress–dilatancy theory and the desire to produce the most simple expressions in equations (14)–(17) which could offer a useful correlation with the data. Statistical optimization has therefore not been attempted. Nevertheless, the maximum departure of the data is 2°, and more than 80% of the data lies within 1° of the correlation. The zone of greatest departure is at low relative density, where there is evidence of greater rates of dilation in plane strain than in triaxial strain, with correspondingly higher angles of shearing. For simplicity, this particular phenomenon was conservatively ignored.

Figure 8 compares the first part of equation (15), which has been shown in Fig. 6 to be almost indistinguishable in operation from Rowe's stress–dilatancy theory, with the data from plane strain tests. The closeness of fit is excellent, the widest departure being 2° in a ϕ'_{max} of 47° on dense Leighton Buzzard sand, G, which was the only soil tested in the simple shear apparatus. A departure of less than 1° was more typical.

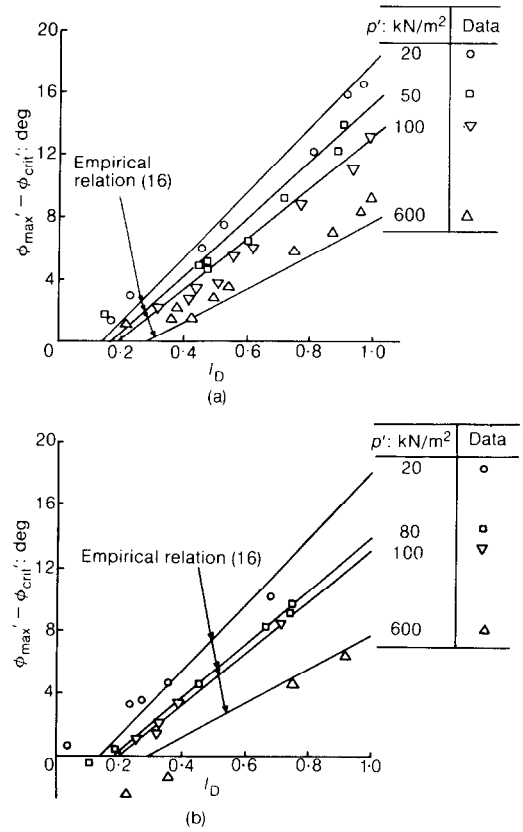

Fig. 9. (a) Triaxial test data for Berlin sand (K) (De Beer, 1965) and (b) triaxial test data for Mol sand (J) (Ladanyi, 1960)

Figure 9(a) traces the triaxial data of Berlin sand and Fig. 9(b) traces Ladanyi's triaxial data (Ladanyi, 1960) for Mol sand, which formed part of De Beer's remarkable study (De Beer, 1965) of the separate effects of relative density and mean effective stress. It will be found that the proposed empirical relations fit the totality of data at various densities and stresses within a typical margin of ±1° and within a maximum margin of ±2°.

Figure 10 records triaxial data of increasing effective stress at failure of sands with initial relative densities either in the vicinity (±0·05) of 0·8 or of 0·5. The chosen axes are similar to those used by Vesic & Clough (1968) in their classic report of high pressure test results, except that advantage has again been taken of the use of $\phi'_{max} - \phi'_{crit}$ as the dilatancy-related strength parameter. It will be seen that the data of $\phi'_{max} - \phi'_{crit}$ in Fig. 10 fall well along the appropriate I_D = constant lines produced using equations (14) and (16), with a typical departure of less

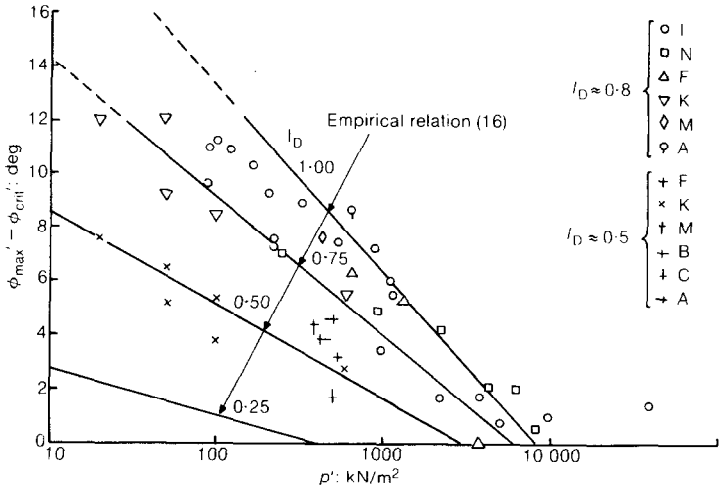

Fig. 10. Triaxial test data for sands in Table 1 failing at various mean effective stresses

than 2°. The logarithmic axis carries p' beyond 10^4 kN/m² into the region where $\pm 2°$ scatter in ϕ'_{max} was commonly observed around the best estimate for ϕ'_{crit}.

The best interpretation of I_R values which are calculated to be negative, owing either to extreme looseness or to high confining stress is that a considerable *contraction* in volume (and consequent increase in I_D) will take place before ϕ'_{crit} is mobilized at $I_R = 0$ after large shear strains. Expression (14) gives an indication of the value of p' (p'_{crit}, say) at which $I_R = 0$ and dilation is suppressed for any initial relative density: In $p'_{crit} = 10 - 1/I_D$. However, a perusal of Fig. 10 will indicate that significant errors may arise in the prediction of p'_{crit} due to the logarithmic scale.

There is some evidence in the literature that certain uniform, rounded sands are little affected by confining pressures less than about 1000 kN/m², after which they begin to crush relatively swiftly. This is shown in Fig. 11(a) where the triaxial data of Lee & Seed (1967) for subangular Sacramento river sand follow the empirical relation (16) in contrast with Ottowa sand which was found not to crush significantly up to pressures of 4000 kN/m² and whose data lie up to 5° above the empirical prediction in the medium-to-high stress range. Fig. 11(b) for Karlsruhe sand

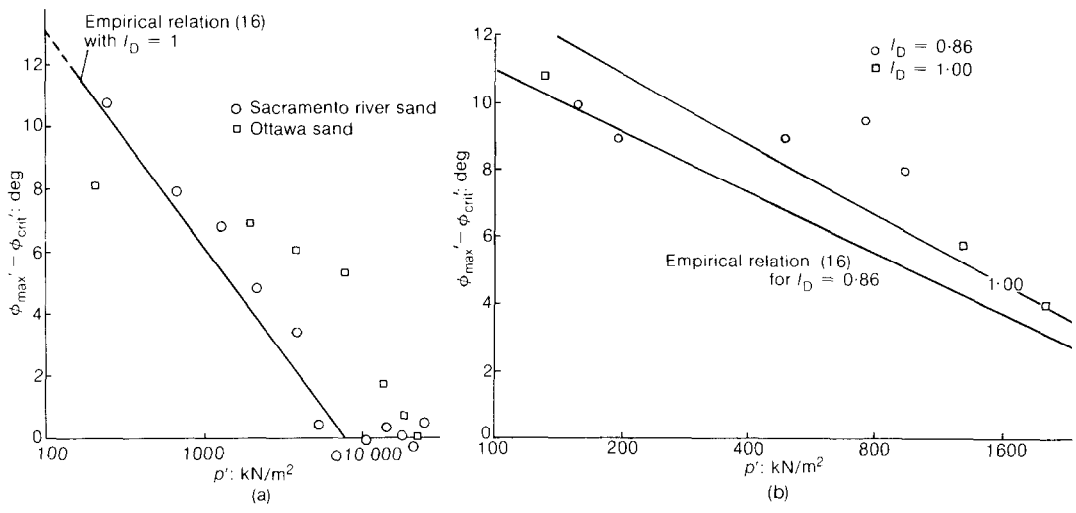

Fig. 11. (a) Triaxial test data for two dense sands at elevated stresses (Lee & Seed, 1967) and (b) triaxial test data for dense Karlsruhe sand (P) at various stresses (Hettler, 1981)

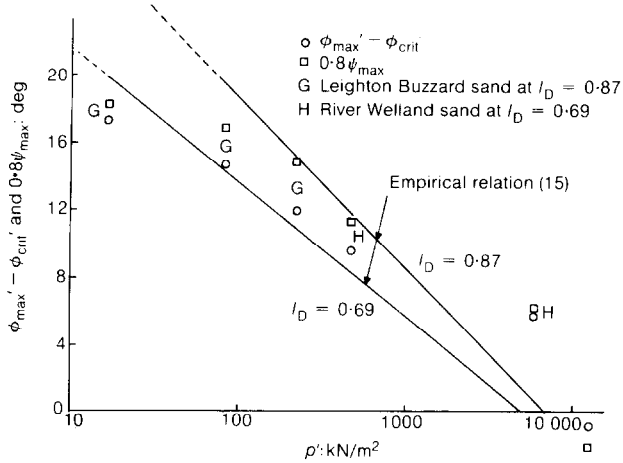

Fig. 12. Plane strain data for sands in Table 1 failing at various mean effective stresses

(Hettler, 1981) demonstrates a similar tendency in a uniform rounded sand noted for its particle strength (Drescher & Vardoulakis, 1982).

Figure 12 introduces the rather sparse available information of plane strain tests conducted at various stresses. Results in the middle band of stresses were broadly as expected, but the low stress test on sand G in the simple shear apparatus and one of the high stress tests on sand H in a plane compression apparatus deviated quite markedly in opposite senses. Both tests must have been difficult to carry out satisfactorily, for contrasting reasons.

For the low stress shear test, the effects of any stress and strain non-uniformities could have been strong. The sand was exceptionally brittle and therefore subject to progressive rupture as ϕ' could drop from at least 52° to 35°. Furthermore, the accurate measurement of stresses as small as 15 kN/m^2 presents severe difficulties. If the result can be relied on, however, it might indicate an absolute limit to the rate of dilation of a dense sand irrespective of how small the stresses are. Until good low stress data are available for a particular sand, it would be prudent to set a limit of 4 to the magnitude of the relative dilatancy index I_R, irrespective of larger values calculated by equation (14). This would have the effect of limiting $\phi'_{max} - \phi'_{crit}$ to 20° in plane strain and 12° in triaxial strain, values which have been attained but not exceeded in the test programmes recalled here.

For the plane compression test on soil H at 6000 kN/m^2, the unexpectedly high rate of dilatancy at failure was matched by an even swifter rate of softening to a critical state: the required boundary displacement was about one half a particle diameter. This suggests that the sand was in a somewhat unstable state within its test chamber, and may correspond to the remarks made earlier about the sudden degradation of some uniform sands which had displayed relatively little progressive crushing. Whether or not the extra mid-range strength was due to this, or to some other cause, it might be prudent to disregard it for design purposes.

It should be recalled that the bulk of data so far presented derive from compression tests with $H/D = 2$ and some undetermined platen friction. Previous studies are ambiguous with regard to the effect of reducing friction on samples with $H/D = 2$. Bishop & Green (1965), in an exhaustive study of Ham sand, conclude that the effect is negligible. Rowe & Barden (1964) found that, across a full range of densities for Mersey sand, ϕ'_{max} measured with lubricated platens on samples with $H/D = 1$ were about 2·5° less than values measured with conventional platens on samples with $H/D = 2$. Tests conducted on both dense and loose Karlsruhe sand with lubricated platens on samples with $H/D = 1/3$ indicated (Drescher & Vardoulakis, 1982) that ϕ'_{max} was about 3° less than in the conventional test. Since the majority of engineers are used to working with test data derived in the conventional way these discrepancies are, perhaps, of interest principally to the research worker.

No attempt has been made here to extend the discussion to tests in triaxial extension, or to tests with arbitrary intermediate stress ratios. These are habitually found to offer strengths that are intermediate to triaxial compression and plane strain values, the differences between which have been shown here to relate solely to dilatancy

effects in the soil. It may be sufficient to note that the soil grains in a triaxial test have considerably greater freedom to deviate laterally than those in a plane strain test, and that in these circumstances the enhancement of strength due to dilatancy might have been expected to be smaller.

IMPLICATIONS FOR LABORATORY SOIL TESTING

The concept of a relative dilatancy index I_R which is coupled with rates of dilatancy, and related to the *extra* angle of shearing above ϕ'_{crit}, has proved powerful in organizing the data of 17 contrasting sands from around the world tested independently by 12 groups of workers. The particular numerical values quoted in expressions (14)–(17) have been capable of retrospectively fitting the conventional data of shearing resistance in the 100–1000 kN/m^2 range of mean effective stress with a typical departure of 1° or 2° once a good value for ϕ'_{crit} has been obtained. The data themselves make clear that a further 1° or 2° error in ϕ'_{max} might be incurred if ϕ'_{crit} were estimated from the shearing strength of a single very loose sample. Cornforth (1973) has reported that the angle of repose of a loosely tipped heap of dry material subjected to excavation of the toe provided a simple bench test for ϕ'_{crit} which should similarly offer an accuracy of about 1°. The likely error in simply taking $\phi'_{crit} = 33°$ for a sand which has been identified as comprising chiefly quartz grains is apparently also only of the order of 1° or 2°.

Although the limiting behaviour of sands in the range 10^3–10^4 kN/m^2 has also been fitted, the uncertainties are greater. Samples deform and contract substantially under test, and the relevance of initial relative density must be questioned, as must the relevance of the strength parameter itself since the shear strains required to mobilize ϕ'_{crit} are large (Vesic & Clough, 1968). The analogy with the engineering problems posed by soft clays in the small stress range is appealing. This suggests that further study of the behaviour of sands under extreme pressures in terms of their instantaneous (e, ϕ', p') states is necessary if more accurate and complete predictions are required. The present correlations underestimate the strengths of some uniform well-rounded sands in this stress range: these materials do not suffer progressive crushing but tend to degrade rather suddenly at some very large stress.

The sands presented in this study are mainly of quartz or feldspar. The presence of substantial proportions of mica, calcite or other materials would be bound to affect both ϕ'_{crit} and the crushing which reduces I_R at high stresses. Billam's (1972) triaxial data for the dilatancy and strength of initially dense samples of granulated chalk, crushed anthracite and limestone sand indicate that they each gain roughly 7° of shearing resistance per tenfold stress reduction, according precisely with the gradient of the $I_D = 1$ line in Fig. 10. What was shown, however, was that reducing the crushing strength of the grains reduced the critical mean effective stress at which dilatancy was suppressed. This implies that parameter Q in equation (12) should be reduced for soils of weaker grains. Billam's data, assuming that they relate to 'soils' with $I_D = 1$, is roughly consistent with the values in Table 2.

Table 2

Q	Grain type
10	Quartz and feldspar
8	Limestone
7	Anthracite
5·5	Chalk

IMPLICATIONS FOR LABORATORY REPORTS

A minimal laboratory report could comprise

(*a*) source of material and assessment of mineralogy
(*b*) grading curve
(*c*) e_{max} and e_{min} by a standard method
(*d*) angle of repose of dry heap
(*e*) in situ density.

Depending on the significance of the problem in hand, the following additional triaxial compression tests taken in order would confirm the relevance or need for recalibration of the empirical relations proposed above

(*f*) loose soil at a cell pressure *circa* 500 kN/m^2
(*g*) dense soil at a cell pressure *circa* 150 kN/m^2
(*h*) dense soil at the highest available cell pressure
(*i*) other relevant tests.

It should then be possible, with a degree of confidence that is proportional to the number of tests conducted, to predict the peak strength of any element of the soil whose density and stress is known.

An alternative consistent approach would be to test a single soil sample from each zone at a density that is lower than any which could reasonably be anticipated in the field and at a confining pressure that is larger than any which could be experienced. The resulting secant value of ϕ'_{max} could then be used conservatively in design for the whole zone, provided that progressive failure could not occur. If there were sig-

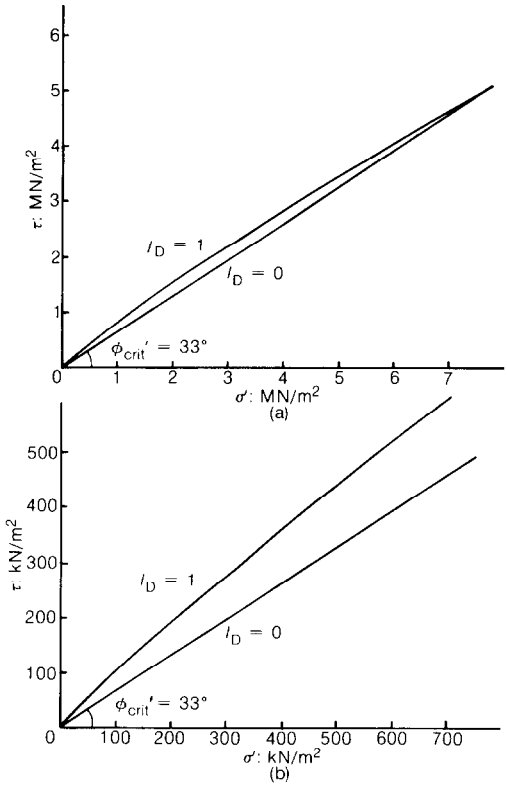

Fig. 13. Mohr envelopes derived from empirical relations for triaxial tests on quartz sands

nificant doubts about the loosest state in the field or the possibility of strain softening, the use of ϕ'_{crit} would be necessary. In this case, the testing programme could simply consist of items (d) and (f).

If required, the 'intrinsic curve' on a τ–σ' diagram can be generated using equation (14) with either equation (15) or (16) depending on whether a plane strain or triaxial envelope is required. It may be assumed, with sufficient accuracy, that the point of contact of a Mohr circle with the intrinsic curve is $\sigma' = p'$, $\tau = p' \tan \phi'_{max}$. Depending on the required scale, curves of the form of Fig. 13(a) or Fig. 13(b) can then be produced for any given I_D. It is then easy to confirm that the conventional tangent parameters (c', ϕ') cannot be fitted without serious misrepresentation. At least three pairs of such values, corresponding to low stress, medium stress and high stress ranges, would be necessary to capture the data of sand at *each* relative density. The fitting of common tangents to Mohr circles must therefore be regarded as a quite inferior method of presentation of soil strength results. Furthermore, the confusion generated by describing any such tangent angle as the 'angle of internal friction' must be deplored as vigorously as the description of the intercept c' as the 'true cohesion'. The detailed studies brought together here tell an unambiguous story of friction and dilation related to secant ϕ' values and ψ values determined separately for each soil test.

IMPLICATIONS FOR FIELD TESTING

Although the foregoing has been concerned with the collation of laboratory test data, it has a strong implication for the interpretation of field tests. The new correlations make clear that there should *not* be a one-to-one correspondence between the strength of granular soils and their relative density, even at a given stress level. A particular angle of shearing resistance (or penetrometer reading) could indicate, for example, either a dense quartz sand with low ϕ'_{crit} and high ψ_{max} values or a less dense felspathic sand with higher ϕ'_{crit} and low ψ_{max} values. The potential confusion between these alternatives might be most significant in a prediction of earthquake-induced excess porewater pressures. The relative density has been shown to correlate only with the dilatant strength component, which can only be resolved from the peak strength (or penetrometer reading) if ϕ'_{crit} is separately determined using a disturbed sample, for example.

It may therefore be considered preferable to measure the angle of dilation ψ_{max} directly, using a self-boring pressuremeter after the fashion of Hughes, Wroth & Windle (1977). The direct correlation in equation (15) between ψ_{max} and I_D at a given stress level should then lead to a more reliable estimate of relative density than would otherwise be available. The attempt to relate cyclic pore pressure generation directly to the measured angle of dilation of the soil (Vaid, Byrne & Hughes, 1981) should prove equally fruitful in studies of liquefaction susceptibility.

The empirical relations (14)–(17) provide a consistent treatment of stress level effects by which the data of all penetration tests can be normalized. Previous treatments of the problem of curved strength envelopes (Baligh, 1976) may thereby be recast as problems of ϕ'_{max} varying with stress (De Beer, 1970), and appropriate methods of calculation developed. The prospects for a consistent approach to reductions in N_q with depth or N_γ with foundation width are then evident. The first objective should be to use the new relations to predict idealized cone penetration resistance profiles which could be compared with those obtained in practice (Mitchell & Lunne, 1978), so that laboratory and field data can be harmonized.

CONCLUSIONS

It has been demonstrated that secant angles of shearing are required in a rational approach to the strength and dilatancy of sands. $\phi'_{max} - \phi'_{crit}$ has been shown to be a useful measure of the extra component of strength due to dilatancy in a dense soil, and an expression has been derived which is operationally indistinguishable from Rowe's well-established theory.

A new relative dilatancy index I_R has been defined in terms of relative density and effective stress level. Extremely simple correlations between I_R and $\phi'_{max} - \phi'_{crit}$, ψ_{max} and $(d\varepsilon_v/d\varepsilon_1)_{max}$ have been supported using the data of 17 soils available in the literature. A corollary is that there is *not* a one-to-one correspondence between ϕ'_{max} and relative density, even at a given stress level, since the dilatancy-related component of strength is $\phi'_{max} - \phi'_{crit}$ and ϕ'_{crit} is a function of mineralogy. Typical values for ϕ'_{crit} are 33° for a quartz sand and 40° for a felspathic sand, though simple laboratory tests should be used as a routine measure. It would be prudent not to invoke I_R values in excess of 4 without test corroboration of the extreme dilatant strength which would be implied. All ϕ'_{max} values in excess of ϕ'_{crit} are transitory with larger strains and subject in design calculations to an assurance that progressive failure will not occur. Negative values of I_R indicate that large contractile strains will occur before ϕ'_{crit} can be mobilized.

The possibility now exists of rationalizing the various *ad hoc* stress level corrections which have emerged in penetration testing and pile design. Clarifications of the relative proportions of friction and dilatant strength components are likely to be particularly important in investigations of the liquefaction potential.

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