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MICRO MECHANICS OF ELASTIC SOIL

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ABSTRACT

This paper presents a review of the stress dependence for soil stiffness at very small strains. Previously published data for sands and clays are presented, and it is shown that in all cases, provided voids ratio is kept approximately constant, then the very small strain stiffness of soils is found to vary with mean effective stress p as $p^{1/2}$. The $p^{1/2}$ dependence of stiffness has long been established for more idealised aggregates comprising regular arrays of spherical particles, and published micro mechanical explanations for this behaviour are presented. A simple mean field approach based on Hertzian contact theory predicts that the dependence should be $p^{1/3}$, but highlights two possible reasons for the apparent discrepancy comparing with available data: (i) contacts may not be Hertzian and (ii) the number of contacts may increase with increasing stress level at approximately constant voids ratio. Two alternative previously published explanations for the $p^{1/2}$ dependence relate to conical contacts between particles and particle chain buckling mechanisms. These mechanisms are presented and discussed, and the paper shows that the $p^{1/2}$ dependence could arise due to one or other of these mechanisms, but not both simultaneously. It seems possible that in densely compacted or overconsolidated soils where voids ratio is approximately constant until yield occurs, contacts may be aspherical and the number of contacts may simultaneously increase with increasing confining stress. In this case the conical contact and particle chain buckling mechanisms are not viable: a more rigorous analysis based on the contact of rough particles is required. It is proposed that such an analysis should allow for the simultaneous elastic squeeze down of surface asperities and increase in the number of asperity contacts under increasing confining stress.

Key words: elasticity, micro mechanics, statistical analysis, stiffness, very small strains (**IGC:** D5)

INTRODUCTION

The stiffness of soil can only be assumed elastic and linear at very small strains ($<10^{-5}$) (Atkinson, 1993). Volumetric and shear effects are then fully uncoupled so that isotropic elasticity may be applied (Viggiani and Atkinson, 1995). The shear modulus of soil at very small strains is easily determined using dynamic methods (Viggiani and Atkinson, 1995) and so the bulk modulus can be easily be related to shear modulus via a Poisson ratio. In general, data shows that at very small strains, soil stiffness increases with effective confining pressure p as $p^{1/2}$ (Houlsby and Wroth, 1991) (note that in this paper all stresses are assumed to be effective stresses), provided voids ratio is kept approximately constant. For densely compacted or overconsolidated materials, voids ratio will be approximately constant until yield occurs. For sands, Wroth and Houlsby (1985) proposed the following relation between shear modulus G and mean effective stress p:

$$\frac{G}{p_r}=A\left(\frac{p}{p_r}\right)^N \tag{1}$$

where p_r is a reference pressure and A and N vary depending on the choice of reference pressure and current strain. Experiments have shown (Wroth et al., 1979) that for both dynamic and static tests on sands, N varies from about 0.5 at very small strains to 1.0 at large strains, corresponding to a fully frictional flow with major rearrangement of particles. For clays, Houlsby and Wroth (1991) suggested that the variation of shear modulus G with effective pressure and overconsolidation ratio could be expressed as:

$$G=\left(\frac{G}{p}\right)_{nc} p^{(1-N)} p_o^N \tag{2}$$

where $(G/p)_{nc}$ is a constant for a given clay and p_o is the preconsolidation pressure. For soils with a given preconsolidation pressure, the voids ratio is approximately constant. Houlsby and Wroth (1991) suggest a value for N of about 0.5 at very small strains, with $G=G_o$, the shear modulus at very small strains.

Dynamic testing methods are used commonly to obtain reliable measurements of the very small strain shear modulus G_o. Hardin and Black (1968) used the resonant column technique to measure the vibration modulus of normally consolidated clay, and found the modulus to increase with confining pressure p as:

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$$G_o = F(e)p^{1/2} \tag{3}$$

where $F(e)$ is a function of the voids ratio e and increases with a decrease in the value of e. Hardin and Black (1966) found the same form of pressure dependence to apply to sands, for both static and dynamic tests. Again, the $p^{1/2}$ dependence of stiffness is evident, if e is kept constant. Chang et al. (1989) quote several empirical relations which relate the average co-ordination number to the voids ratio of an aggregate. In this case, for soil at a given effective pressure, the increase in stiffness with a reduction in voids ratio can be seen to be due to the increase in the number of particle contacts.

The very small strain stiffness of soils may alternatively be determined by measuring the velocity of shear waves through a triaxial sample using the bender element method (Viggiani and Atkinson, 1995). Viggiani and Atkinson (1995) proposed the following equation for the very small strain shear modulus G_o based on data for reconstituted samples of speswhite kaolin:

$$\frac{G_o}{p_r} = A\left(\frac{p}{p_r}\right)^{N} (\mathrm{OCR})^{M} \tag{4}$$

where OCR is the overconsolidation ratio and p_r is a reference pressure to make (4) dimensionally consistent (p_r, which influences the value of A, is normally taken to be 1 kPa or atmospheric pressure). Equation (4) could alternatively be written:

$$\frac{G_o}{p_r} = A\left(\frac{p}{p_r}\right)^{N\text{-}M}\left(\frac{p_o}{p_r}\right)^{M} \tag{5}$$

where p_o the preconsolidation pressure. If p_o is kept constant the voids ratio of a given soil with a given preconsolidation pressure can be assumed to be approximately constant. In this case the pressure dependence in (5) at approximately constant voids ratio is $p^{N\text{-}M}$. The values for N and M for kaolin were found by Viggiani and Atkinson (1995) to be 0.653 and 0.196 respectively, giving an overall pressure dependence of $p^{0.46}$. For reconstituted London clay, the value of $(N\text{-}M)$ was calculated to be 0.51. Thus the stiffness is once again seen to be proportional to $p^{1/2}$. Viggiani and Atkinson (1995) found this pressure dependence to be true for both undisturbed and reconstituted samples, demonstrating that G_o is unaffected by the soil structure and fabric. This implies that at very small strains, the deformation is due to elastic deformation at points of contact between particles. Biarez and Hicher (1994) suggest a general law which may be applied as a first approximation to the majority of soils at very small strains:

$$\frac{E_e}{\sqrt{p}} = \frac{450}{e} \tag{6}$$

where e is the voids ratio, p is the mean effective stress in MPa and E_e is an elastic secant modulus measured in MPa and defined as $E_e = q/\varepsilon_{11}$, where q is the deviatoric stress in MPa and ε_{11} is the major principal strain. The law is based on data for sands, silts, marls and clays. It is evident, then that the very small strain stiffness of soils varies with mean effective stress p as $p^{1/2}$, provided voids ratio is constant. In order to explain this power-law dependence of elastic modulus on confining pressure, a micro mechanical view of soil is required. To explain the existence of the 0.5 power, it is necessary to consider the mechanics of contact between elastic soil grains. Of course soil grains vary greatly in size and shape and the aggregations of particles are highly complex. Most recent methods of quantifying soil microstructure have used the fractal concept (Moore and Donaldson, 1995; Vallejo, 1995). However, in this paper we perform a simple mean-field estimate for the dependence of elastic modulus on pressure for simple isotropic stress states, on the assumption that particles are spherical, and modify the analysis for the case of conical contact between particles. The first step is to consider a single contact between two smooth, non-conforming surfaces. We first examine simple Hertzian contact between two spheres and apply a mean field approach to obtain aggregate stiffness as a function of stress level for isotropic conditions. The mean field approach highlights two possible reasons for the discrepancy between predicted and observed soil behaviour. A modified approach is therefore used to explain the $p^{1/2}$ dependence for stiffness. Two previously published explanations are used, based on alternative single and multiple contact mechanisms. However, it seems possible that in real soil aspherical contacts may occur together with an increase in the number of particle contacts under increasing stress at approximately constant voids ratio. This paper shows that in this case, a more rigorous analysis for rough spheres would be required, in which the distribution of asperities on each particle surface should be considered.

HERTZIAN THEORY

We will consider, for simplicity, the contact of two identical spheres (the more general case of contact between any two smooth non-conforming surfaces can be found in Johnson (1985)). For two spheres each of radius R with an area of contact of radius a_h, Hertz (1882a) gives the mutual approach δ of the spheres as a function of the applied normal load P:

$$\delta = \frac{2a_h^2}{R} = \left(\frac{3P(1-\nu_p^2)}{E_p\sqrt{(2R)}}\right)^{2/3} \tag{7}$$

where E_p and ν_p are the Young's modulus and Poisson ratio of the particle material respectively. The contact stiffness E_c can be defined as P/δ and is readily seen to increase with applied load P as $P^{1/3}$. We might write

$$E_c \propto P^{1/3} E_p^{2/3} \tag{8}$$

In order to see how this variation of contact stiffness with load affects the dependence of the elastic modulus of an aggregate on applied pressure, we apply a mean-field approach to an array of spheres, each identical to the spheres considered above.

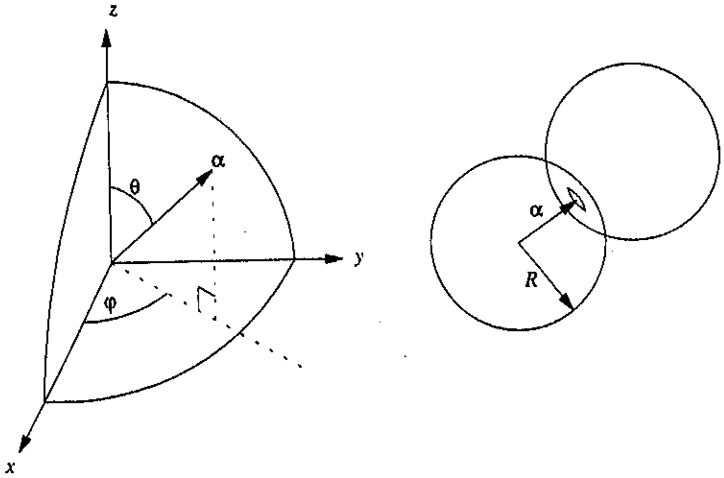

Fig. 1. Spherical co-ordinate system

MEAN FIELD ESTIMATE OF STRESS-STRAIN RESPONSE

We consider the response in isotropic compression of a triaxial sample containing a random array of identical spheres. The detailed analysis can be found in Jenkins and Strack (1993), and is only summarised here. The end result is presented for isotropic stress states, in order to elucidate the possible reasons for the dependence of stiffness on mean effective pressure p as $p^{1/2}$. The spheres are each of radius R, and spherical co-ordinates are chosen, shown in Fig. 1 with respect to a fixed Cartesian system. The z-axis is taken to be in the direction of the axis of the triaxial sample. For a general pair of particles in contact (Fig. 1), α is taken to be the unit vector from the centre of a sphere to the contact at its surface. It is assumed that the particle centres move in accordance with a uniform displacement field. The displacement u of a contact relative to the centre of a sphere is calculated from the average strain ε of the aggregate (taking compressive strain as positive):

$$u_i = -R\varepsilon_{ij}\alpha_j \tag{9}$$

The underlying assumption in this statement is that the average particle spin is equal to the rigid body rotation of the aggregate (Jenkins and Strack, 1993; Chang et al., 1989). The force $F(\alpha)$ exerted on a sphere at a contact with orientation α is given in terms of its components parallel and perpendicular to α:

$$F_i = -P\alpha_i + T_i \tag{10}$$

with $T.\alpha = 0$. The key step in the mean field approach (a full description of the mean field approach is given in Jenkins and Strack, 1993) is the application of the principle of virtual work, with an arbitrary set of compatible strains and displacements:

$$\sigma_{ij}\varepsilon_{ij} = N_s \iint F_i u_i A(\alpha) d\omega \tag{11}$$

where σ_{ij} is the stress tensor, N_s is the number of spheres per unit volume and $A(\alpha)$ is the orientational distribution of contacts defined such that $A(\alpha)d\omega$ is the probable number of contacts in an element of solid angle $d\omega$ centred at α. For an isotropic distribution of contacts,

$$A(\alpha) = \frac{C}{4\pi} \tag{12}$$

where C is average number of contacts per particle, called the co-ordination number. The end result of the mean field approach for isotropic conditions is that the volumetric strain ε_v is given by the equation:

$$\varepsilon_v \propto \left(\frac{p}{G_p}\right)^{2/3} \left(\frac{v}{C}\right)^{2/3} \tag{13}$$

where p is the mean effective stress, G_p is the shear modulus of the particle material and v is the specific volume. Hence the bulk modulus of the aggregate K_v, given as p/ε_v (and hence shear modulus) is related to pressure p by:

$$K_v \propto p^{1/3} G_p^{2/3} C^{2/3} \tag{14}$$

if voids ratio is kept approximately constant.

The mean-field approach described above predicts that the stiffness of the aggregate, for purely elastic deformations, increases with pressure p as $p^{1/3}$. If the number of particle contacts is constant, then (14) is of the same form as (8). The form of pressure dependence in (14) is therefore a direct consequence of the assumption that Hertzian theory applies. An apparent discrepancy exists between the 0.5 power of p found to influence stiffness empirically and that developed in the mean-field theory (14). However, the mean field analysis has been useful since the final result in (14) offers two possible explanations for the discrepancy:

(i) Hertzian theory may be an incorrect approach for determining the deformations at particle contacts
(ii) Co-ordination number may be a function of strain

These two possible mechanisms are now investigated. It is interesting to note that (13) predicts that if Hertzian theory applies, then in order to produce an agreement with experimental data, co-ordination number C should increase with volumetric strain ε_v as $\varepsilon_v^{1/2}$.

SINGLE AND MULTIPLE CONTACT MECHANISMS

Goddard (1990) suggested that the discrepancy between the $p^{1/3}$ dependence in (14) and the $p^{1/2}$ dependence found in much of the experimental data might be due to aspherical contacts between particles. He analysed the contact between a plane and a sphere of radius R with a pointed (i.e. conical) asperity of shallow included angle π-$2\alpha_c$ (with $2\alpha_c << \pi$), shown in Fig. 2. Goddard (1990) derived the following expressions for the contact stiffness, E_c, defined as $dP/d\delta$:

$$E_c=(32\hat{\mu}P/\pi\alpha_c)^{1/2}, \quad \text{for } 0<P\leq P^* \tag{15}$$

$$E_c=(96\hat{\mu}^2RP)^{1/3}, \quad \text{for } P>P^* \tag{16}$$

where the transition force P^* is given by

$$P^*=\frac{9}{32}\pi^3\hat{\mu}R^2\alpha_c^3 \tag{17}$$

and $\hat{\mu}$ is an elastic constant, related to the Lamê constants and Poisson's ratio for the particle material. For the contact between a sphere and a conical asperity, as opposed to the contact between a plane and a conical asperity, it is simply necessary to replace R by $R/2$ in the above equations. It is evident from the mean-field analysis given in the previous section, (15) will give rise to the following relation between the bulk modulus of the aggregate and mean pressure:

$$K_v \propto p^{1/2}G_p^{1/2} \tag{18}$$

assuming that voids ratio and average co-ordination number remain constant. The transition force arising from elastic squeeze-down of the conical contact gives rise to a transition pressure in the aggregate, beyond which all contacts are Hertzian and the elastic modulus of the aggregate varies with pressure as $p^{1/3}$. Goddard (1990) points out that this is consistent with data produced by Duffy and Mindlin (1957) for elastic wave velocities in face-centred cubic (FCC) packings of high-tolerance steel balls. However, Duffy and Mindlin (1957) found a corresponding aggregate of low tolerance steel balls to exhibit no such transition, and the pressure dependence was of the form $p^{1/2}$ throughout the range of experimental pressures used. Goddard (1990) proposed an alternative mechanism by which the $p^{1/2}$ dependence of stiffness in soils may occur. He considered in an array of particles under compression, sample-spanning chains of particles capable of supporting axial compression, as is observed in discrete element models of elastic grains (Cundall and Strack, 1979). If there is insufficient lateral force from neighbouring particles due to a deficiency of contacts, then Euler buckling of the particle chain will occur until it is prevented by the formation of new contacts and the development of sufficient lateral force. Consider a pair of particles in a particle chain, each of radius R and which

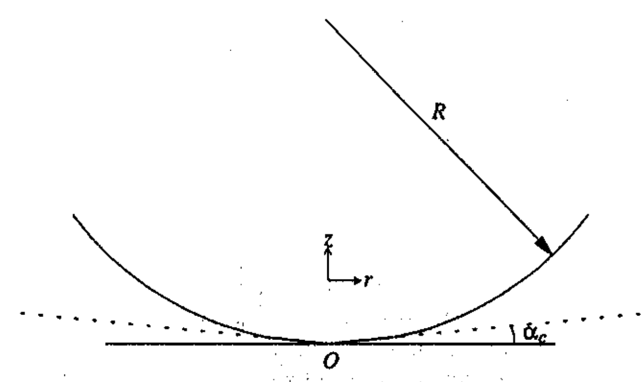

Fig. 2. Axisymmetric contact between a plane and a conical asperity

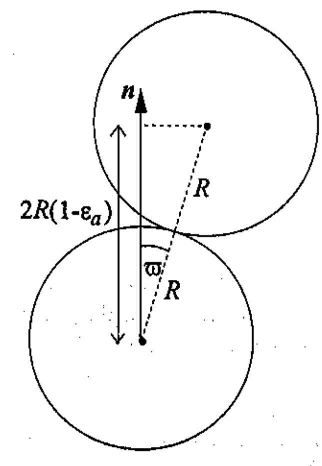

Fig. 3. Small rotation at a particle contact

execute a small rotation $\bar{\omega}$ normal to their line of centres (Fig. 3). If the direction of the axis of the original chain is $\boldsymbol{n}$, then the axial compressive strain ε_a in the column of particles is given by:

$$\varepsilon_a=\varepsilon_{ij}n_in_j \tag{19}$$

It is readily seen that after a small rotation $\bar{\omega}$, the distance between particle centres now has a projection of $2R(1\text{-}\varepsilon_a)$ along $\boldsymbol{n}$ (Fig. 3). Thus

$$2R(1-\varepsilon_a)=2R\cos\bar{\omega} \tag{20}$$

which for small $\bar{\omega}$ reduces to

$$\bar{\omega}^2=2\varepsilon_a \tag{21}$$

The infinitesimal lateral displacement η of one particle relative to another is

$$\eta=2R\bar{\omega}=R\sqrt{8\varepsilon_a} \tag{22}$$

The way in which this lateral motion gives rise to new contacts depends on the gap distribution in the array. Goddard (1990) assumed the probable number of new contacts $dC(\xi, \boldsymbol{n})$ formed by a lateral movement through a distance between ξ and $\xi+d\xi$ perpendicular to $\boldsymbol{n}$ to be governed by a uniform distribution over some interval $(0, l)$. In this case the total number of new contacts formed by the lateral displacement η is given by:

$$\Delta C/\Delta C_l = R\sqrt{8\varepsilon_a}/l, \quad \eta \leq l \tag{23}$$

$$\Delta C/\Delta C_l = 1, \quad \eta \geq l \tag{24}$$

where l is a function of n and ΔC_l denotes the maximum number of new contacts which can be formed. Goddard (1990) states that any smooth distribution function for $C(\xi, n)$ would lead to a result of the form (23, 24) for small ε_a. By averaging over all n, the total number of contacts is seen to increase with global strain magnitude as $|\varepsilon|^{1/2}$ until the maximum contact density is achieved. This will give rise to a transition pressure, below which the elastic modulus of the aggregate increases with pressure according to (18), and above which the $p^{1/3}$ dependence predicted by Hertzian theory applies.

DISCUSSION

The transition pressure which emerges in the conical contact model increases with increasing value of α_c (Eq.(17)) i.e. increasing asperity height. An array of low tolerance spheres (Duffy and Mindlin, 1957) of diameter 1/3 inch $\pm 50\times10^{-6}$ inches did not experience such a transition over the range of pressures used, but a similar array of spheres of tolerance $\pm 10^{-5}$ inches was found to exhibit this transition. It is conceivable that for soil particles, the conical asperity model may apply, and that rough soil particles might never reach such a transition pressure under the range of confining pressures normally used in the triaxial cell. However for irregular particles such as soil particles, the exact definition of α_c would be difficult. Furthermore, experiments by Oda (1977) and others cited by Chang et al. (1989) indicate a one-to-one relationship between mean co-ordination number and voids ratio. All of the data discussed in the Introduction shows that if the effect of voids ratio is approximately constant, then the elastic modulus of the aggregate is dependent solely on the confining pressure p and increases with p as $p^{1/2}$. If constant voids ratio implies constant mean co-ordination number (and therefore constant total number of particle contacts), then the conical contact model offers a plausible explanation for the $p^{1/2}$ dependence.

If Hertzian theory does apply to the contact of rough soil particles (which requires that the loads transmitted across particle contacts are sufficiently large (Greenwood and Tripp, 1967)), then the $p^{1/2}$ dependence can be explained in terms of the generation of new contacts. This requires that co-ordination number can increase with increasing compressive volumetric strain whilst voids ratio can be assumed to be approximately constant. An examination of the data by Oda (1977) shows that although the data could be described adequately by a relationship between voids ratio and co-ordination number, there is sufficient scatter in the data to show that at a given voids ratio, the co-ordination number might not be uniquely defined. This is also true of data cited by Chang et al. (1989). Furthermore, the mean co-ordination numbers calculated by Oda (1977) related to compacted glass balls under zero applied stress, and no tests were performed for a densely compacted or overconsolidated array of balls under increasing stress levels, where voids ratio could be assumed to be constant. It is therefore conceivable that co-ordination number may increase with increasing confining stress, whilst the voids ratio remains approximately unchanged. Ko and Scott (1967) examined the compression of regular arrays of spheres such as face-centred cubic (FCC) and body-centred cubic (BCC), and developed a 'holey' model in which some particles initially carry no load due to being slightly smaller than the other particles. Thus for a regular array in which particle shapes are imperfect, the elastic closure of gaps leads to an aggregate stiffness which increases with pressure p more rapidly than $p^{1/3}$. However, if gaps between particles close simply due to an increase in the normal approach between grains, then in general, the co-ordination number C increases with volumetric strain ε_v as (Goddard, 1990):

$$\Delta C \propto \varepsilon_v \tag{25}$$

Substituting (25) in (13) gives the following dependence for aggregate modulus K_v on pressure p:

$$K_v \propto p^{0.6} G_p^{0.4} \tag{26}$$

In order to obtain the $p^{1/2}$ dependence for the elastic modulus of such a regular array of spheres obeying Hertzian theory at particle contacts, it is necessary to resort to the idea of particle chain buckling and the infinitesimal rotation at contacts to provide sufficient lateral force. The particle chain buckling model gives rise to a transition pressure beyond which $p^{1/3}$ dependence applies. Goddard (1990) points out that this is consistent with data for sands under prolonged vibration (Drnevich and Richart, 1970). However this phenomenon might be described as inelastic stiffening due to shake-down, and the data for the compression of regular arrays of high tolerance spheres (Duffy and Mindlin, 1957) is certainly supported better by the conical contact model. Goddard (1990) concedes that particle chain buckling is, strictly speaking, irreversible. It is therefore not applicable to the purely elastic deformation of a soil aggregate.

It is interesting to note that Goddard (1990) presents two very different models: one is based on the idea that Hertz does not apply at individual contacts, and the number of contacts remains constant. The other model assumes that Hertz applies, and that the number of contacts increases with increasing pressure. The question arises as to what happens if the conical contact model applies, and the number of contacts increases according to the particle chain buckling mechanism. Clearly the $p^{1/2}$ dependence is lost: the mean field approach readily gives the bulk modulus of the aggregate as a function of p:

$$K_v \propto p^{0.6} G_p^{0.4} \tag{27}$$

The two mechanisms offered by Goddard (1990) should therefore be seen as alternatives, if the very small strain behaviour of soil is to be explained. If the confining stress is sufficiently low to avoid elastic squeeze down of contacts, then the conical contact model offers a plausible explanation if new particle contacts are not generated. If the confining stress is sufficiently high to invoke Hertzian behaviour at all contacts, then the $p^{1/2}$ dependence can only be explained by the generation of new contacts with negligible change in voids ratio: that is if small changes in e can have a significant effect on co-ordination number. However, the particle chain buckling mechanism is irreversible and therefore not strictly applicable to elastic soil: an alternative multiple contact mechanism is required. It may be that neither the conical contact model nor the particle chain buckling model apply, and that a statistical approach based on the contact of rough surfaces (Greenwood and Tripp, 1967; Archard, 1957; Greenwood and Williamson, 1966) might be more appropriate. In this case the analysis should take into account the distribution of surface asperities on the soil particles in the aggregate, so that under increasing confining stress, elastic squeeze down of asperities is possible together with the generation of new asperity contacts.

CONCLUSIONS

A review of the very small strain behaviour of soil under isotropic stress conditions has been presented. It has been found that for strains $<10^{-5}$, data for both sands and clays show that the stiffness of soil varies with stress level p as $p^{1/2}$, provided voids ratio is approximately constant. A mean-field approach has been used with Hertzian contact theory to predict the dependence of stiffness on stress level, and it has been shown that the predicted dependence is $p^{1/3}$. However, the mean field approach suggests two possible reasons for this discrepancy: either contacts are not Hertzian, or the number of particle contacts may increase with increasing stress level. Two alternative previously published mechanisms for $p^{1/2}$ dependence have been examined: a single conical contact mechanism and a multiple contact particle chain buckling mechanism. Applying a mean field approach, each mechanism independently predicts the $p^{1/2}$ dependence for stiffness. However, the mechanisms should be seen as alternatives, since if both occur simultaneously, the $p^{1/2}$ dependence is lost. Furthermore, the particle chain buckling mechanism is irreversible, and therefore strictly speaking, not applicable to elastic soil. If the confining stress is sufficiently low that conical contacts are possible, then the conical asperity mechanism offers a plausible explanation if new contacts are not generated under increasing stress levels. If the confining stress is sufficiently high to invoke Hertzian behaviour at all contacts, then the $p^{1/2}$ dependence can only be explained if it is possible to generate new contacts under increasing stress with negligible reduction in voids ratio. It seems possible that in overconsolidated or densely compacted soils, where voids ratio is approximately constant under increasing stress until yield occurs, particle contacts may be aspherical, whilst the number of contacts may also increase under increasing confining stress. A more rigorous analysis would then be required, in which the distribution of surface asperities on each particle should be considered, so that under increasing confining stress, elastic squeeze down of asperities is possible together with the generation of new asperity contacts.

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