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Monte-Carlo simulations of the time dependent failure of bundles of parallel fibres

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ABSTRACT. – Monte-Carlo simulations have been used successfully to predict the lifetimes and the variability in the lifetimes of bundles of parallel elements. Good agreement has been obtained between the simulations and lifetimes extrapolated from empirical results for parallel-lay ropes. An increase in the scatter of the elements' lifetimes reduces that of the bundle; similarly, reducing the scatter could significantly increase the lifetime of the bundle.

Nomenclature

A_i	Cross-sectional area of an element i
CV	Coefficient of variation
$E[t]$	Mean value of t
$H(t)$	Cumulative distribution function for lifetime of element
k	Boltzmann's constant
ℓ	Normalised bundle load
l_q	Load carried by a surviving element q
L	Constant load of a bundle
m	Number of sub-bundles in a rope
n	Number of parallel fibres in a sub-bundle
s	Weibull shape parameter
t	Time
T	Absolute temperature
$U(\cdot)$	Thermal activation energy
U_0	Stress-free activation energy
$\tilde{U}_0$	A constant related to stress-free activation energy
α	A constant related to exponential-law breakdown rule
β	A parameter related to Gumbel distribution
γ	A constant related to exponential-law breakdown rule
δ	A parameter related to Gumbel distribution
η	A Weibull constant

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θ	Power of power-law breakdown rule
$\kappa(\cdot)$	Breakdown rule
μ_{tc}	Mean lifetime of a sub-bundle
ν	Activation volume
σ	Stress
σ_ℓ	Constant stress
$\sigma(t)$	Stress at time t
$\tilde{\sigma}$	A constant related to theoretical bond strength
$\sigma_{n,j}$	Stress of the i-th element of a bundle of n elements
σ_{tc}	Standard deviation of the lifetime of a sub-bundle
τ	Mean time between failure events
τ_0	Period of bond vibration
ϕ	A constant related to power-law breakdown rule
ω	Weibull scale parameter
$\mathcal{E}_t$	Bundle lifetime efficiency
$\Gamma(\cdot)$	Standard gamma function
$\Lambda(\cdot)$	Cumulative distribution function of a normalised uniform variate
$\Psi(\cdot)$	Shape function

1. Introduction

An important characteristic of a well designed structure is its ability to support the applied loads during its operating lifetime. Many materials which survive initial applied loads well below their ultimate load fail after a period of time. Creep-rupture (failure under sustained load) is therefore a major concern for engineers, particularly when dealing with materials, such as prestressing tendons, which sustain high permanent loads.

Parallel-lay ropes, made from synthetic materials, are being used increasingly as prestressing tendons in the civil engineering industry. The ability to predict the lifetimes of the ropes under stress is therefore very important, since the ropes are made from materials which exhibit a creep-rupture phenomenon.

Models to predict the lifetimes of bundles of parallel elements (*e.g.* parallel-lay ropes) have been proposed by Phoenix [1978a; 1978b; 1979a], and Smith & Phoenix [1981]. A statistical process known as the quantile process was used to model the bundles; it was assumed that the lifetime distributions of the bundle elements follow the conventional Weibull distribution and they concluded that the distribution of the times-to-failure of the bundles made from a large number of elements is Gaussian. The Weibull distribution was chosen to represent the elements' lifetime because of mathematical difficulties associated with the use of other distributions with the quantile process [P, 1979a; S & P, 1981].

The asymptotic results obtained by Phoenix and Smith are believed to underestimate the bundle strength [S & P, 1981] and because of mathematical complexities, the exact results for bundles with a few or a moderate number of elements were ignored. Despite these models, the failure process of bundles of parallel elements is not well understood and therefore studies into the lifetime behaviour of such bundles are needed to successfully predict their creep-rupture.

The strength and the lifetime distributions of twisted yarns, the basic elements of parallel-lay ropes, are argued to follow the Gumbel distribution [Smith, 1982] since the constituents of the yarns, the fibres, are numerous and are not bonded with a matrix. This argument is supported by the findings of Amaniampong & Burgoyne [1994], and Amaniampong [1992] that the Gumbel distribution best represents the aramid yarn strength. Thus the lifetime distributions of yarns do not always obey the Weibull distribution.

In this paper, a short review of the bundle theories about the creep-rupture (time-to-failure) of the bundles of parallel elements is presented. A Monte-Carlo study of the lifetime of bundles of parallel elements is conducted. Both Weibull and Gumbell distributions are used to represent the lifetimes of the elements. This allows a study of the differences in the times-to-failure of the bundles brought about by the different statistical distributions. Monte-Carlo simulations are also used to check the convergence of Phoenix's asymptotic results.

An attempt is also made to model the time-to-failure of parallel-lay ropes from the filament. Empirical data for aramid filaments obtained from the literature is used in the model and the results are compared with creep-rupture data of Parafil Type G ropes which have aramid yarns as the core material.

1.1. Definitions

In this paper, reference will be made to a number of different levels in hierarchy of rope construction, and the terms used for these are often not consistent in the literature. To avoid ambiguity, they will be defined here.

The smallest indivisible element is the *filament*, typically produced by a single hole in a spinneret or die; in the case of the aramid fibre Kevlar, the filaments have a diameter of about 2 microns.

Filaments are not usually used individually in rope construction; they are brought together to form *yarns* immediately after spinning. Depending on final use and filament diameter, the number of filaments may vary between a few hundred and a few thousand. Yarns usually have a degree of twist, which has great influence on the final properties, but can be supplied untwisted. There is an optimum twist level that maximises the strength of the yarn; less twist results in a lower strength due to bundle effects; more twist reduces strength since the fibres are no longer axial.

For *parallel-lay ropes,* a large number of yarns are brought together to form the rope, with no additional twist being introduced. Tests on *Parafil ropes* are referred to in this work, which are made by Linear Composites Ltd; the Type G ropes are made from 1000 filament yarns of Kevlar 49 aramid fibres, twisted to give the optimum strength.

The term *fibre* will be used to refer to a generic component of a *bundle.* A yarn can be regarded as a bundle of filaments, and a rope as a bundle of yarns.

Associated with each level of rope structure will be a *characteristic length.* This can be regarded as the length which is affected by a broken element. For an untwisted parallel-lay rope, this length will be of the order of a few metres, while for twisted yarn, it may be as small as a fraction of a millimetre.

2. Models for predicting the lifetime of bundles of parallel elements

2.1. Creep-rupture lifetime of single filaments

The general form of the probability distribution function for the lifetime of single filaments (elements) under constant load with stress history $\sigma(t)$, $t \geq 0$ is assumed to be [Coleman, 1958]

$$H(t;\, \sigma) = 1 - \exp\left\{-\Psi\left[\int_0^t \kappa(\sigma(t))\, dt\right]\right\}, \quad t \geq 0 \tag{1}$$

where $\kappa(\cdot)$ and $\Psi(\cdot)$ are known as the breakdown rule and the shape function respectively. $\kappa(\cdot)$ also represents the rate parameter associated with the failure events of molecules which follow the exponential distribution [Kausch, 1978]. The time-to-failure of many filaments can be adequately represented by the Weibull distribution [P, 1978a] so the shape function, Ψ, is commonly assumed to be of the Weibull form

$$\Psi(x) = \eta x^s, \qquad x \geq 0 \tag{2}$$

where s and η are positive constants. The constant η is a non-dimensional parameter representing the volume of the filament.

The forms commonly assumed for κ are the power-law breakdown rule,

$$\kappa(\sigma) = \phi\sigma^{\theta}, \qquad \sigma \geq 0 \tag{3}$$

and the exponential-law breakdown rule,

$$\kappa(\sigma) = \alpha \exp(\gamma\sigma), \qquad \sigma \geq 0 \tag{4}$$

where ϕ, θ, α and γ are positive constants.

For creep-rupture, there a constant stress history, *i.e.*, $\sigma(t) = \sigma_\ell$ for $t \geq 0$. Combining Eqs. (1) and (2) and using the constant stress history gives

$$H(t) = 1 - \exp\{-\eta\,[\kappa(\sigma_\ell)]^s\, t^s\}, \quad t \geq 0 \tag{5}$$

The distribution of the filament lifetime may be written in the conventional Weibull form

$$H(t) = 1 - \exp\left\{-\left(\frac{t}{\omega}\right)^s\right\} \tag{6}$$

where the Weibull scale parameter, ω, is given as

$$\omega = \frac{\exp(-\gamma\sigma_\ell)}{\alpha\eta^{1/s}} \tag{7}$$

for the exponential-law breakdown rule, and

$$\omega = \frac{1}{\sigma_\ell^\theta \, \eta^{1/s} \, \phi} \tag{8}$$

for the power-law breakdown rule.

The mean, $E[t]$, and the coefficient of variation, CV, of the lifetime of the filaments under constant load or stress are given as

$$E[t] = \frac{\Gamma\left(1 + \frac{1}{s}\right)}{\eta^{1/s} \, \kappa(\sigma_\ell)} \tag{9}$$

$$\mathrm{CV} = \sqrt{\frac{\Gamma\left(1 + \frac{2}{s}\right)}{\Gamma^2\left(1 + \frac{1}{s}\right)}} \tag{10}$$

where $\Gamma(\cdot)$ is the standard gamma function. The power-law breakdown rule gives a linear relationship between $\ln(E[t])$ and $\ln(\sigma_\ell)$ with slope $-\theta$, but the exponential-law breakdown rule leads to a linear relationship between $\ln(E[t])$ and σ_ℓ with slope $-\gamma$.

The mean time, τ, between failure events for a given molecule under constant stress, $\sigma_\ell \geq 0$, is given by the molecular theory of absolute reaction rates as [K, 1978]

$$\tau(\sigma_\ell) = \tau_0 \, \exp\left[\frac{U(\sigma_\ell)}{k\,T}\right] \tag{11}$$

where τ_0 is the period of bond vibration, $U(\sigma_\ell)$ is the thermal activation energy required for the event at applied stress, σ_ℓ, T is the absolute temperature, and k is Boltzmann's constant. The time until such an event occurs is a random variable which follows the exponential distribution with a rate parameter $\kappa(\sigma_\ell) = 1/\tau(\sigma_\ell)$ [K, 1978]. Thus, if $h(t)$ is the density function for the failure event then,

$$h(t) = \kappa \, \exp(-\kappa \, t) \tag{12}$$

The function $U(\sigma_\ell)$ can be approximated by the linear function [Eyring, 1936; Zhurkov, 1965]

$$U(\sigma_\ell) \approx U_0 - \nu\sigma_\ell \tag{13}$$

where ν is the activation volume and U_0 is the stress-free activation energy. Phoenix & Tierney [1983] argued that a logarithmic approximation to the function, $U(\sigma_\ell)$, fits better than the linear one and suggested the relation

$$U(\sigma_\ell) \approx -\tilde{U}_0 \ln\left(\frac{\sigma_\ell}{\tilde{\sigma}_0}\right) \tag{14}$$

where $\tilde{U}_0$ is a positive constant related to the stress-free activation energy and $\tilde{\sigma}_0$ is a positive constant related to the theoretical bond strength.

The power-law breakdown rule Eq. (3) arises when Eqs. (11) and (14) are combined, with $1/\kappa$ replacing τ so that

$$\phi = \frac{1}{\tau_0\, \tilde{\sigma}_0^{\tilde{U}_0/k\,T}} \tag{15}$$

and

$$\theta = \frac{\tilde{U}_0}{k\,T} \tag{16}$$

The exponential-law breakdown rule (Eq. (4)) results from the combination of Eqs. (11) and (13), and replacing τ with $1/\kappa$ whereby

$$\alpha = \tau_0^{-1} \exp\left[-\frac{U_0}{k\,T}\right] \tag{17}$$

and

$$\gamma = \frac{\nu}{k\,T} \tag{18}$$

2.2. Lifetime of bundles of parallel elements under constant load

The lifetime of bundles of parallel elements (*e.g.* parallel-lay ropes) under constant load is asymptotically normally distributed, if the elements' time-to-failure under constant stress (or load) follows the Weibull distribution [P, 1978a; 1978b, 1979a; S & P, 1981].

For a bundle of n parallel elements under a constant load, $L = n\ell$, the cumulative distribution function for the lifetime of the elements (Eq. (5)) becomes

$$H(t) = 1 - \exp\{-\eta\, [\kappa_2(\ell)]^s\, t^s\} \tag{19}$$

Here κ_2 is a breakdown rule which is associated with the load instead of the stress. κ_2 has the same form as κ (Eqs. (3) and (4)) but with the parameters ϕ_2 and γ_2 replacing ϕ and γ.

If $H(t) = G(\kappa_2\, t)$, where $G(z)$, $z > 0$ is a continuous function with $G(0) = 0$ and $\int_0^\infty e^z\, y\, dG(y) < \infty$ for $z > 0$, Then $G(y)$ is given as

$$G(y) = 1 - \exp\{-\eta\, y^s\} \tag{20}$$

If

$$\Omega(x, \psi) = \begin{cases} \dfrac{1}{\left[\kappa_2\left(\dfrac{\psi}{1-x}\right)\right]}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases} \tag{21}$$

is bounded and has a continuous derivative on [0, 1], and $g(t)$, $0 \leq t < 1$ is the right-continuous inverse function of $G(y)$ for $y \geq 0$, *i.e.* $\inf\{y;\ G(y) > t\}$, then the mean, μ_{tc}, and the variance, σ_{tc}^2/n, for the lifetime of the bundle of n parallel elements are given by [S & P, 1981]

$$\mu_{tc} = -\int_0^1 \Omega'(t,\,\ell)\, g(t)\, dt \tag{22}$$

$$\sigma_{tc}^2 = 2\int_0^1 s\,\Omega'(s,\,\ell)\, dg(s) \int_0^t (1-t)\,\Omega'(t,\,\ell)\, dg(t)$$

where $\Omega'(x,\,\psi) = \dfrac{d\Omega(x,\,\psi)}{dx}$.

3. Monte-Carlo study of the lifetime of bundles of parallel elements

In the rest of this paper, the lifetime of bundles of parallel elements under a constant load is modelled with the Monte-Carlo method. For a specific applied load, the time-to-failure of the element with the lowest lifetime is evaluated (generated). After the element's failure, the load is redistributed among the surviving elements by a specific rule and more elements may fail. This process continues until the last element fails and the bundle's lifetime is the time elapsed until the rupture of the last element.

The process is involved and requires complicated mathematical formulation but the Monte-Carlo simulations allow the study of the behaviour with relative ease when the complexity makes it too difficult to formulate an answer analytically [A & B, 1992]. The main components of the prodecure are discussed in the subsequent sections.

Since parallel-lay ropes are made from yarns, the creep-rupture behaviour of the ropes are modelled from the yarns. The yarn characteristics are first generated from the filaments by using the series-parallel model [S, 1982]. The procedure is described in detail in Section 3.3.2. The lifetime characteristics of the yarns are generated from the filaments because data on the creep-rupture of bare yarns is not available. This is, however, likely to introduce some errors since the effect of twist on the creep-rupture is not well understood.

3.1. Random number generation

For a bundle of n elements, n random numbers are generated to characterise their lifetimes. It is assumed that the lifetime distribution of the elements is governed by the Weibull distribution and the exponential-law breakdown rule given by Eqs. (6) to (8), or the Gumbel distribution. The cumulative distribution function of the Gumbel distribution is given as

$$H(t) = 1 - \exp\left[-\exp\left\{\frac{(t-\beta)}{\delta}\right\}\right] \tag{23}$$

where β and δ are constants.

The random numbers from the Weibull and the Gumbel distributions are obtained from the inverse transform method by equating probabilities [Rubinstein, 1981]. Thus, if $\Lambda(\overline{\tau})$ is the probability associated with $\overline{\tau}$ from a normalised uniform distribution, then $\Lambda(\overline{\tau}) = H(t) = \overline{\tau}$. The normalised uniform random numbers are generated by the multiplicative congruential method [*see*, R, 1981].

From Equation (6) a Weibull random number, t, is obtained as

$$t = \omega\,[-\ln(1-\Lambda)]^{1/s} \tag{24}$$

Since $1-\Lambda$ is distributed in the same way as Λ, t can also be written as

$$t = \omega\,[-\ln\Lambda]^{1/s} \tag{25}$$

Similarly, a random number, t, from the Gumbel distribution is obtained from Eq. (23) as

$$t = \delta\ln[-\ln\Lambda] + \beta \tag{26}$$

3.2. Detailed assumptions of the model

A system of n parallel elements under a constant load is considered. It is assumed that the elements exhibit creep-rupture behaviour and fail randomly as time passes. It is also assumed that the elements come from the same population so that the lifetimes can be represented by a parametric distribution (Weibull or Gumbel distribution). At any specific time the surviving elements share load according to a specific rule. Two main load sharing rules are considered:

(i) The load is shared among the surviving elements according to the *equal load sharing rule* (ELS rule) and the elements are assumed to have the same cross-sectional area. The implication is that surviving elements have both the same stress and carry the same load at any specific time. This assumption allows comparison with the asymptotic results by S & P [1981] which were formulated under similar assumptions.

(ii) *Equal stress sharing rule* (ESS rule): The cross-sectional area of the elements varies and follows the Gaussian distribution; at each stage of the process, surviving elements carry loads which are proportional to the ratio of their areas to the total area of the surviving elements.

Under case (i), if σ is the initial stress of the system and $j < n$ elements have failed prior to a specific time, the stress of each surviving $n-j$ elements, $\sigma_{n,j}$, is given as

$$\sigma_{n,j} = \frac{\sigma}{1-\dfrac{j}{n}} \tag{27}$$

Under case (ii), if L is the load applied to the system and $j < n$ elements have failed prior to a specific time, the load, l_q, carried by a surviving element, q, is given as

$$l_q = L\,\frac{A_q}{\displaystyle\sum_{i=n-j}^{n} A_i} \tag{28}$$

where A_q is the cross-sectional area of the element, q. The stress of each surviving element, $\sigma_{n,j}$, becomes

$$\sigma_{n,j} = \frac{L}{\sum_{i=n-j}^{n} A_i} \tag{29}$$

Thus at each stage of the process, stresses are shared equally among the surviving elements.

Let $T_{q,j}$ be the expected lifetime of a surviving element acting alone under stress $\sigma_{n,j}$ and $\Delta t_{q,j}$ be the actual time the element spends under stress, $\sigma_{n,j}$, because of the failure of weaker elements. A basic assumption is that each element consumes a fraction of its lifetime during each portion of the stress history. The following cumulative damage laws are therefore considered:

(*a*) no cumulative damage; only time and stress since the last element's failure is taken into account, so the failure of an element in a particular stage of the process is independent of the previous stress history.

(*b*) linear law cumulative damage; the proportion of the lifetime remaining of the q^{th} element to fail, R_q, is given as

$$R_q = \prod_{j=0}^{q-2} \left(1 - \frac{\Delta t_{q,j}}{T_{q,j}}\right), \qquad q = 2, \ldots, n \tag{30}$$

(*c*) root law cumulative damage;

$$R_q = \prod_{j=0}^{q-2} \left[1 - \left(\frac{\Delta t_{q,j}}{T_{q,j}}\right)^{0.5}\right], \qquad q = 2, \ldots, n \tag{31}$$

(*d*) square law cumulative damage;

$$R_q = \prod_{j=0}^{q-2} \left[1 - \left(\frac{\Delta t_{q,j}}{T_{q,j}}\right)^{2}\right], \qquad q = 2, \ldots, n \tag{32}$$

Thus the time that an element q spends under the stress history $\sigma_{n,q-1}$ before failing becomes $R_q T_{q,q-1}$.

3.3. Monte-Carlo procedure

3.3.1. *Bundles of parallel filaments*

The lifetimes of the bundles of parallel filaments are simulated with the assumption that the filament's times-to-failure follow the Weibull or the Gumbel distributions. These filaments are referred to as Weibull or the Gumbel filaments respectively. For the Weibull filaments, Eqs. (6), (7), and (25) are used to generate the lifetimes directly, but for the

Gumbel filaments, it is assumed that they have the same mean and the standard deviations as the Weibull filaments. The Gumbel parameters, β and δ, are therefore obtained as

$$\beta = \mathrm{E}\,[\mathrm{x}] + 0.5772\,\delta\,\frac{\mathrm{SD}\,[\mathrm{x}]\,\sqrt{6}}{\pi} \tag{33}$$

where E [x] and SD [x] are the mean and the standard deviations obtained from Eq. (10). The resulting values of β and δ can then be used in Eq. (26).

3.3.2. *Parallel-lay ropes*

The times-to-failure of parallel-lay ropes are modelled in two stages. The lifetime characteristics of the yarns are first simulated from the filaments and the yarns are then used to obtain the lifetime behaviour of the ropes.

Filaments to yarns. – The yarns are assumed to consist of a chain of m sub-bundles with n parallel filaments in each sub-bundle (bundles of parallel filaments), in accordance with the series-parallel model [S, 1982]. The lifetimes of the yarn are assumed to be best represented by the Gumbel distribution, and therefore the Gumbel parameters, δ and β, (Eq. (23)) are obtained as [S & P, 1981]

$$\delta = \frac{\sigma_{tc}}{\sqrt{(2\,n \ln m}} \tag{34}$$

and

$$\beta = \mu_{tc} + 0.5\,\delta\,[\ln(\ln m) + \ln(4\,\pi) - 4\ln m] \tag{35}$$

where σ_{tc} and μ_{tc} are the standard deviation and the mean lifetime of a sub-bundle.

For the parallel-lay ropes, Weibull filaments are used to generate yarns whose lifetimes follow the Gumbel distribution (Gumbel yarns). This is because the Weibull distribution has been found to be adequate for representing the filament lifetime distribution of aramids [Wagner *et al.,* 1986]. The Weibull filaments are used to generate the lifetimes of a large number of bundles of parallel filaments at a given characteristic length (sub-bundles). The mean, μ_{tc}, and the standard deviation, σ_{tc}, of the sub-bundles are then evaluated and these values are inserted into Eqs. (34) and (35) to obtain the Gumbel lifetime parameters, β and δ, for the yarn at the required characteristic length of the rope. The process is carried out for a large range of applied loads, and the relationships between the Gumbel parameters and the applied loads (or stresses) are established by using a regression analysis. With the Gumbel lifetime parameters known, the lifetime of the yarns are generated from Eq. (26).

Yarns to ropes. – The process of generating the lifetimes of the bundles of parallel fibres (or parallel-lay ropes which are bundles of parallel yarns) is started by assigning normalised uniform random numbers to the bundle elements. Eq. (26) (for the Gumbel yarns) or Eq (25) (for Weibull filaments) is then used to generate the lifetimes of the elements under a specified applied load. The failure time of the weakest element is recorded as the failure time of that stage. After an element's failure, the applied load is re-distributed among the surviving elements according to the load sharing rule. The ELS rule is used for the parallel-lay ropes but both ELS and ESS rule are tried for the bundles of parallel filaments.

The new lifetimes of the surviving elements under the re-distributed load are then generated by using the corresponding random numbers and Eq. (26) or (25). The time-to-failure of the weakest element at this stage of the process is recorded after applying the required cumulative damage law (Sec. 3.2) as the failure time for the stage. The process continues until all the elements have failed and the failure times for the stages

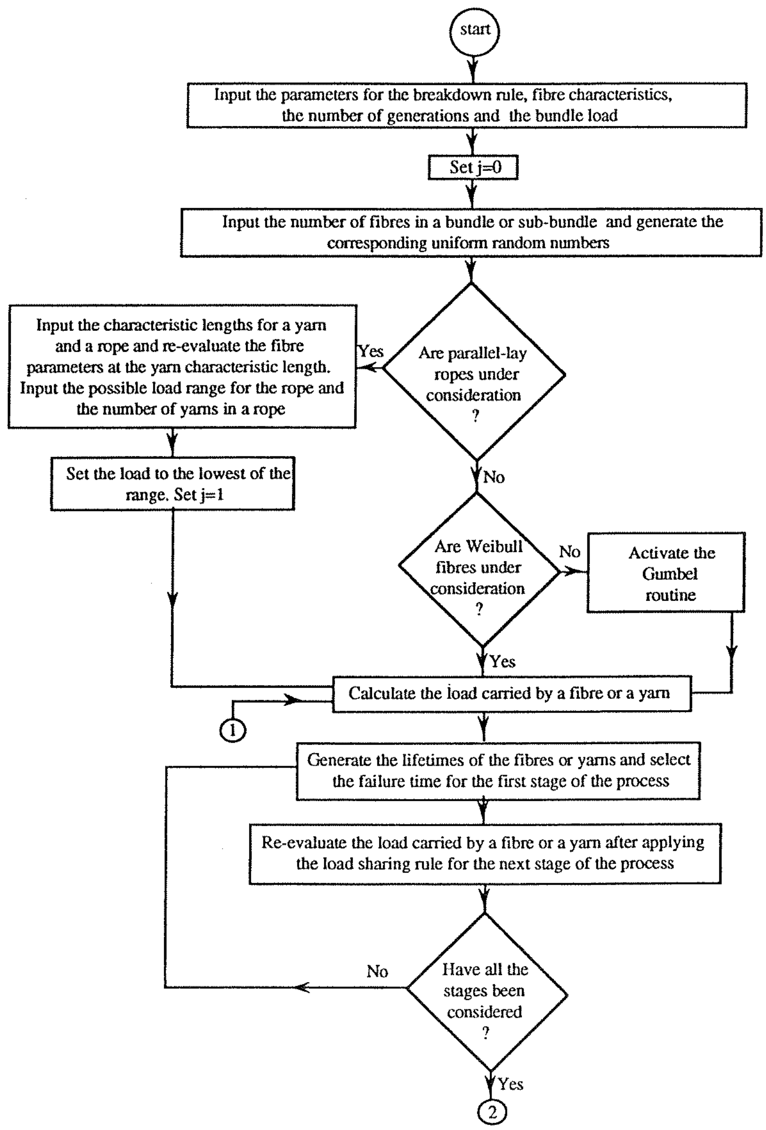


Fig. 1. – Flow chart showing the Monte-Carlo procedure to evaluate the lifetimes of bundles.

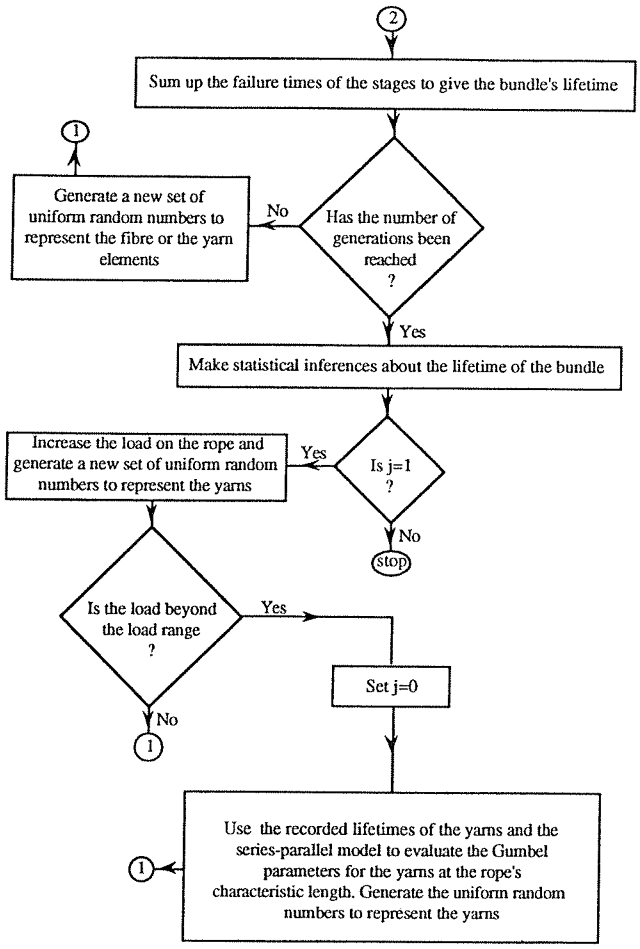


Fig. 1 continued.

are summed to give the bundle's lifetime. The whole process is repeated to generate a large number of bundle lifetimes which can be used with standard statistical methods to make predictions. The procedure is illustrated in Figure 1.

4. Results and discussions

Two different processes are considered in this section. The first process is the simulation of the lifetime for bundles which are made of parallel filaments and the second process is about bundles made from parallel yarns (parallel-lay ropes). All the results discussed below come from one thousand simulations.

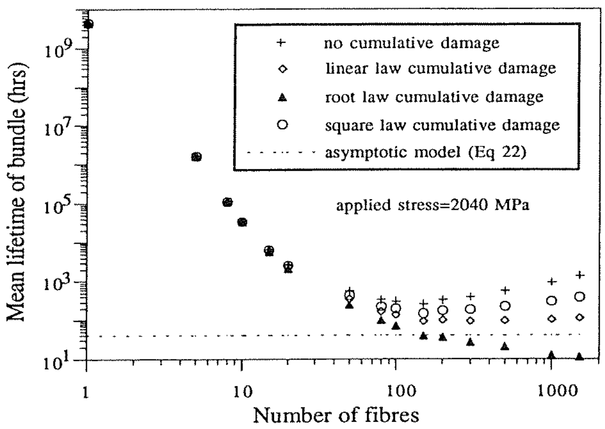
4.1. Bundles of parallel filaments

The simulations of the lifetime process are based on the following sets of parameters which are derived from the experimental results of 50 mm long Kevlar-49 aramid filaments [W *et al.*, 1984; 1986]:

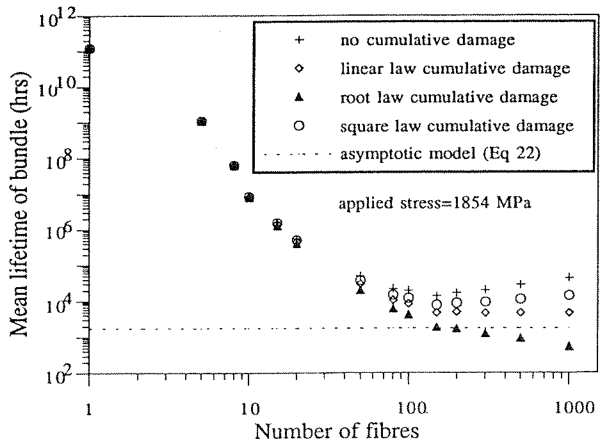
- temperature = 21°C;

- exponential-law breakdown rule parameters (Eq. (4));
  $\gamma = 1.789 \times 10^{-2}$ mm^2/N, $\alpha = 4.105 \times 10^{-24}$ hrs;
- cross-sectional area of filaments;
  mean $= 1.236 \times 10^{-4}$ mm^2, standard deviation $= 0.2394 \times 10^{-4}$ mm^2;
- Weibull lifetime parameters (Eqs. (6) and (7));
  shape parameter, $s = 0.1985$, length parameter, $\eta = 1$;
- Weibull strength parameters;
  'scale parameter = 3400 MPa, shape parameter = 10.3.

Figure 2 shows typical results comparing the lifetimes of bundles for different cumulative damage laws and also shows Smith and Phoenix [1981] asymptotic results (Eq. (22)). Different applied stresses are used for the two cases of the load sharing rules, so the lifetimes from Figures 2(*a*) and 2(*b*) are not comparable. The square law and the



(a) Equal load sharing



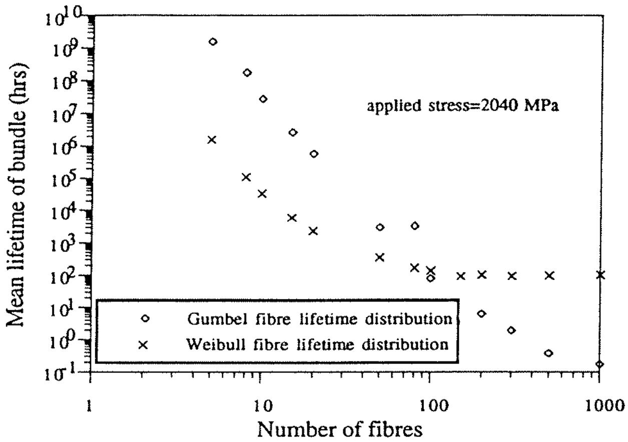
(b) Equal stress sharing

Fig. 2. – Comparison of the lifetimes of bundles for different reduction factors.

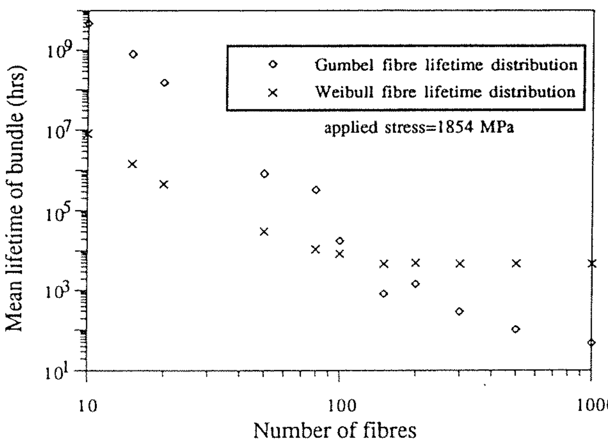
no-cumulative-damage law lead to functions of the size which have a minimum. Thus as the number of elements in the bundle increases, the lifetime first drops, but eventually rises. This behaviour is quite unexpected. The root law gives rise to bundles whose lifetimes are monotonically decreasing functions of the size, whilst the linear law yields bundles whose lifetimes decrease with size and eventually become asymptotic.

The simulations with no cumulative damage give bundles with the longest lifetime followed by the square law and the linear law. The root law gives bundles with the shortest lifetime. The differences in the time-to-failure become apparent only for medium and large sized bundles. In the given results this occurs when the bundle contains more than twenty elements.

It is not expected that lifetimes of the bundles will decrease and later increase with size, neither is it believed that in reality the lifetime is a monotonically decreasing function



(a) Equal load sharing



(b) Equal stress sharing

Fig. 3. – Comparison of results from elements which obey Weibull and Gumbel distributions.

of the size. The linear law for the cumulative damage therefore seems to be closer to reality and the subsequent studies are based on it. In fact, empirical evidence is lacking and, to the authors' knowledge, no work has been reported in the literature about the best cumulative damage law to be used.

In Figure 3, results for bundles (yarns) which are made from fibres (filaments) whose lifetimes follow the Weibull and the Gumbel distributions are compared. The mean and the standard deviations for the lifetimes of the fibres are assumed to be the same for both distributions.

Small bundles with elements whose lifetimes obey the Gumbel distribution survive longer under the same applied stress than those with Weibull elements, but the reverse is true for large bundles. Thus the size effect is less pronounced for bundles with Weibull fibres.

Since the lifetimes of the Kevlar-49 aramid filaments are best described by the Weibull distribution [W *et al.,* 1986; Wu *et al.,* 1988], the subsequent results are based on the Weibull distribution. Three different plotting scales, namely the log-log, the semi-log and the linear plots, are used for the subsequent graphs.

Typical results of the effect of the two load sharing rules, the equal load sharing (ELS) and the equal stress sharing (ESS), on the lifetime of bundles are presented in Figure 4. The size effect is also shown.

The equal stress sharing rule results in bundles with longer lifetimes. However, the effect of the load sharing rule on the lifetimes diminishes as the bundles size increases. For instance, maximum relative differences of 659% and 3% are obtained for 20-element and 1000-element bundles respectively (*Fig.* 4).

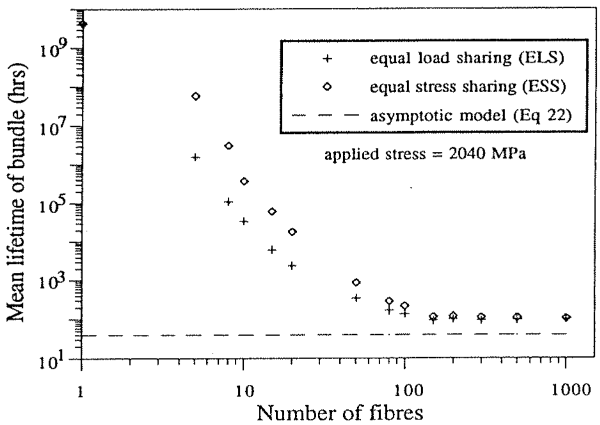


Fig. 4. – Typical results of the lifetimes of bundles under ELS and ESS (Weibull fibres; linear cumulative damage law).

There is a profound size effect associated with the lifetime of the bundle. For example, the mean lifetime of a 1000-element bundle (*Fig.* 4) is about nine orders of magnitude less than that of a single element. This agrees with the approximation by [P, 1978b]. He approximated the bundle lifetime efficiency, $\mathcal{E}_t$, (ratio of the mean lifetime of the bundle to that of an element) by $\mathcal{E}_t \approx (\gamma\sigma)^{-1/s}$ where σ is the stress, s is the Weibull lifetime shape parameter and γ is a parameter of the exponential-law breakdown rule (Eq. (4)). The above expression gives $\mathcal{E}_t = 1.3 \times 10^{-8}$ whilst the corresponding value from the simulation is 2×10^{-8} for the 1000-element bundle.

The lifetime decreases rapidly with increasing size of the bundle. The asymptotic results [S & P, 1981] is too conservative and is worse for bundles with a small or moderate number of elements. For instance, by taking the reference lifetime as the asymptotic value, Figure 6.4 gives relative differences of 768% and 155% for 100-element and 100-element bundles respectively. The convergence to the asymptotic results is very slow and indeed 1000 elements are not enough for its applicability. It is not immediately obvious that the results will ever converge to the asymptotic result. But this depends heavily on the cumulative damage law used. It may be that a 0.9 power law would converge to the asymptotic result.

In Figure 5, typical results of the effect of the scatter in the elements' lifetime on the bundle are presented. The scatter is reflected in the Weibull shape parameter; the variability of the elements' lifetime is inversely proportional to the shape parameter.

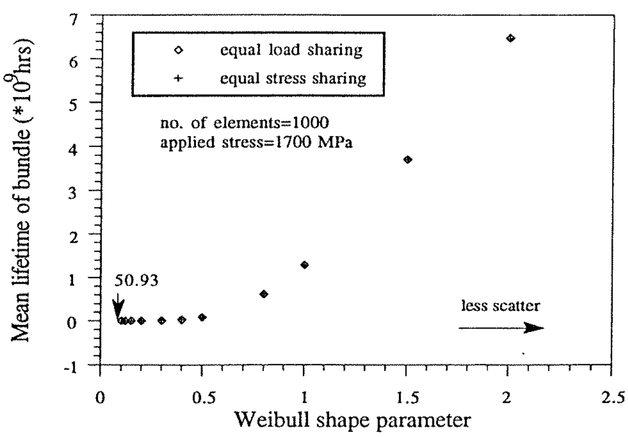


Fig. 5. – Effect of the variability in the lifetime of elements on the lifetime of the bundle (Weibull fibres; linear cumulative damage law).

The lifetimes of the bundle decrease dramatically with increasing scatter in the fibres' lifetimes. This is because the larger the scatter, the more likely it is that a group of elements may fail at an early period, thereby increasing the load on the surviving elements and facilitating the failure process. Thus if the fibres can be made more uniform, the Weibull shape parameter will increase and a very large increase in bundle lifetimes may be possible.

Figure 6 shows the response of the bundle's lifetime to the applied stress. At very high stresses, the bundle fails quickly but the lifetime is substantially improved when moderate stresses are applied. As an example, by dropping the applied stress of a 1000-element bundle from 73% (2380 MPa) to 52% (1700 MPa) of the filament's ultimate tensile stress (UTS), the lifetime increases from 6 minutes (7 minutes) to 12 years (14 years), according to the ELS (ESS) version of the model. The reason for the dramatic increase in the lifetime of the bundle lies in the exponential function which is associated with the failure of the elements.

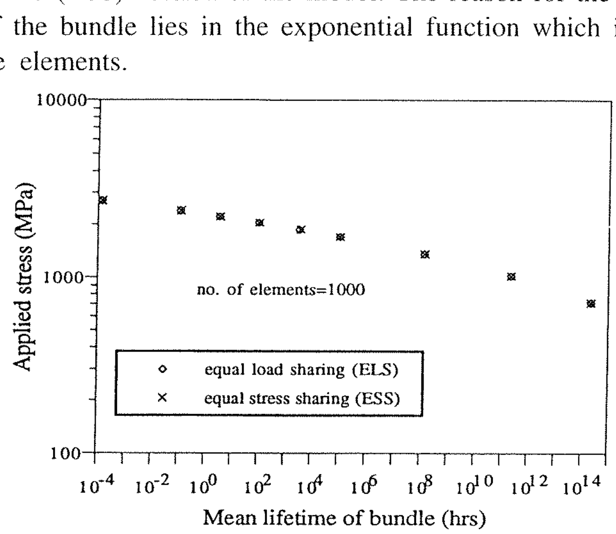


Fig. 6. – Response of the lifetime of bundle to stress (Weibull fibres; linear cumulative damage law).

The effect of the variability of the elements' cross-sectional area on the lifetime of the bundle is presented in Figure 7. This study is only possible for the equal stress sharing (ESS) rule. The larger the scatter in the cross-sectional area, the longer the bundle survives. The effect is significant, however, only for moderate and large scatter. For instance, the increase in the lifetime given in Figure 7 is significant only when the coefficient of variation of the cross-sectional area is 25% or above, which is very unlikely in practice.

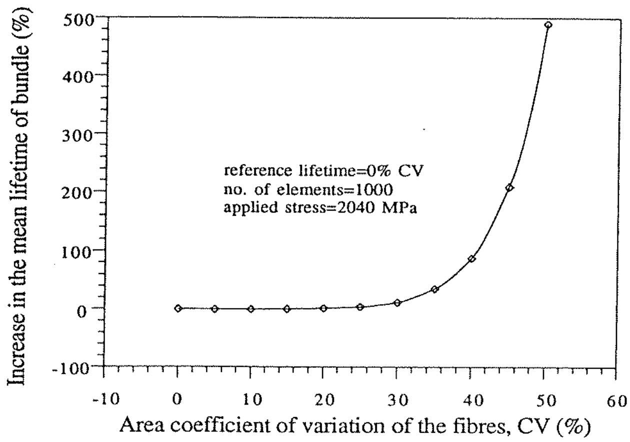


Fig. 7. – Effect of the variation in the cross-sectional area of elements on the lifetime of the bundle (Weibull fibres; linear cumulative damage law).

As an example of the way the process works, Figure 8*a* shows two separate simulations. Stage n of the process represents the time which elapses between the failure of the

$(n-1)$-th element and that of the n-th element. The bundle's failure is triggered by the failure of a small fraction of the elements. in the example, the bundle's failure is brought about by the rupture of about 36% of the elements.

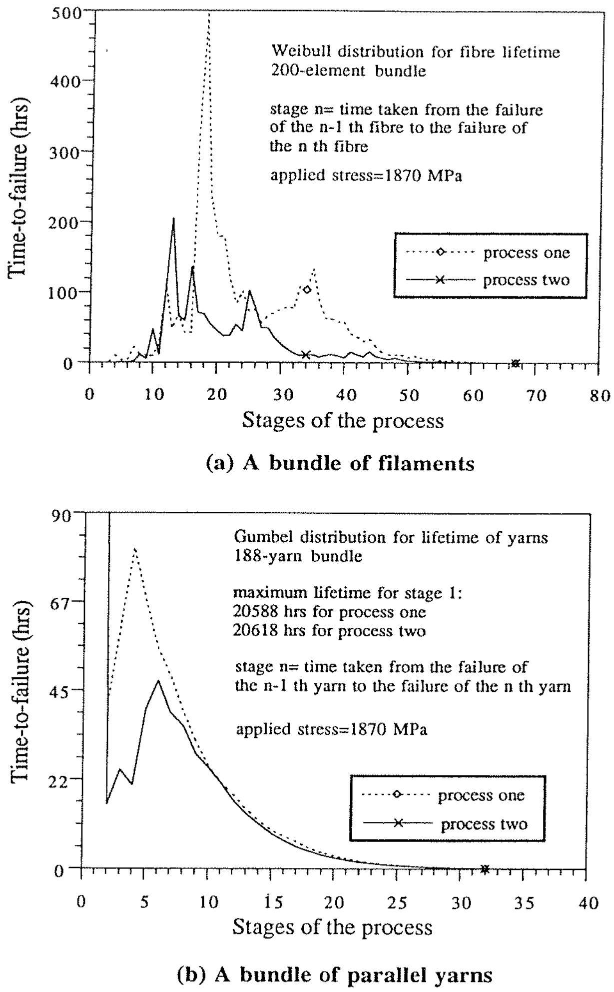


Fig. 8. – Typical failure process of bundles of parallel elements.

4.2. Parallel-lay ropes (bundles of parallel yarns)

The following values are used for the simulations:

- yarn characteristic length, $\delta^* = 0.3575$ mm;
- temperature $= 21°$C;
- length of rope $= 6.2$ m;
- number of fibres per yarn $= 1000$.

The characteristic length of the yarn is the average value obtained from [A, 1992] and the length of the rope is the characteristic length associated with Parafil Type G ropes [Burgoyne & Flory, 1990].

An extrapolation of the lifetime distribution associated with the 50 mm long Kevlar-49 and aramid filaments is required to obtain a corresponding distribution at the length of 0.3575 mm. With the assumption that the parameter η (Eq. (5)) is directly proportional to the specimen length, the extrapolation is possible. However, application of this to the filaments resulted in extraordinarily long-life ropes (*e.g.* the average lifetime of 94-yarn ropes (6.2 m long) at an applied stress of 2040 MPa was 1.0×10^{13} hrs). An alternative is to assume that $\eta = K\, l^{\nu}$ [Watson & Smith, 1985], where ν and K are positive constants. Phoenix *et al.*, [1988] obtained $\nu = 0.6$ for the strength distribution of Kelvar-49 aramid filaments. The use of this value of ν also resulted in ropes with long lifetimes, although shorter than the results from the direct proportionality (*e.g.* the average lifetime of 94-yarn ropes (6.2 m long) at an applied stress of 2040 MPa was 4.4×10^{8} hrs). The conservative approach is therefore adopted in the present simulations; the lifetime is assumed to be independent of length so that the lifetime distributions for both the 50 mm and the 0.3575 mm aramid filaments are the same. In this case, the mean fibre lifetime at an applied stress of 2040 MPa would be 4.4×10^{9} hrs.

Figure 9 shows the relationship between the Gumbel parameters and the applied stress for a 6.2 m yarn. The length of a sub-bundle is taken as 0.3575 mm. The semi-log plot gives a straight line, which is an indication that an exponential relationship exists between the stresses and the Gumbel parameters. The knowledge of this is helpful in any future asymptotic model.

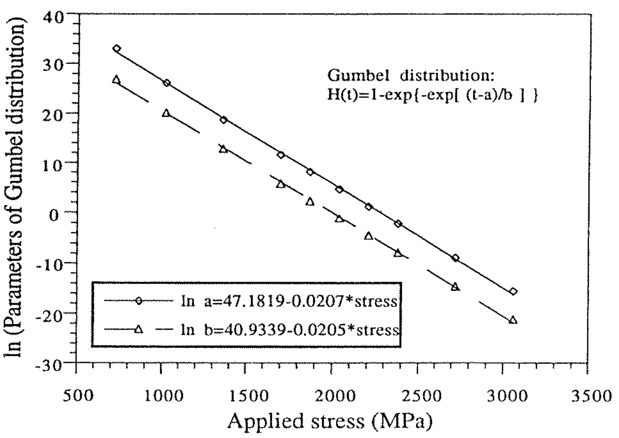


Fig. 9. – Relationship between Gumbel parameters and stress of a 1000–filament Kevlar-49 yarn.

In Figure 8*b* examples of the failure process are presented. Stage n of the process represents the time which elapses between the failure of the $(n-1)$-th element and that of the n-th element. Like the bundles of parallel filaments, the rupture of the rope is triggered by the failure of a small proportion of the yarns. Failure of about 20% of the yarns is enough to cause the immediate rupture of the rope. This has an implication in the rope construction. A rope with a large scatter in the lifetime of the yarns is likely to rupture more quickly than one with yarns with similar lifetime properties. The processes indicate that the lifetime of the first yarn to fail is crucial for the total lifetime of the rope. In the example, the first yarn fails after over 20000 hours, while the maximum duration for each stage of the rest of the process is only about 80 hours.

The effect of size on the bundle's lifetime is presented in Figure 10. Surprisingly, the rope's lifetime increases with increasing size. This is contrary to the behaviour observed for the bundles of parallel filaments. The increase in the lifetime is possibly due to the dispersion associated with the number of yarns in a rope. It is worth noting, however, that the increase in the lifetime is so slight that it may not be observable. The lifetimes of a 47-yarn rope and a 188-yarn rope at the stress of 2040 MPa are 96 and 98 hours respectively. The differences cannot be attributed to noise since the rise is consistent, however for pratical purposes little will be lost in ignoring these differences.

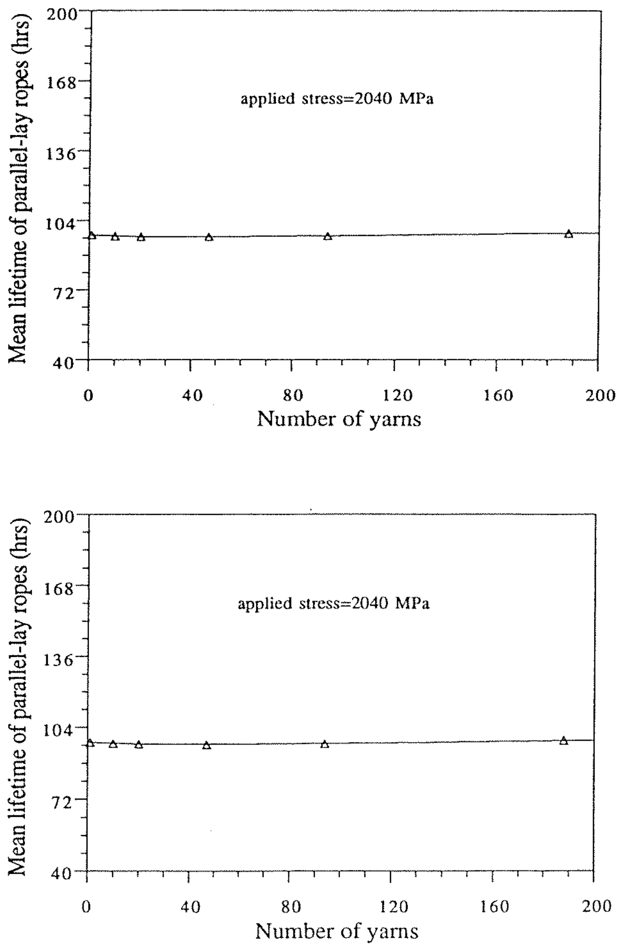
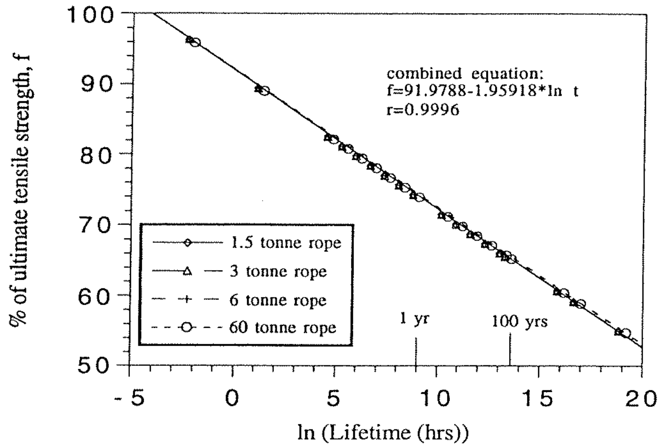


Fig. 10. – Effect of size on the lifetime of Parafil Type G ropes.

Figure 11 shows the relationship between the applied stress and the lifetime of the ropes. The ultimate tensile stresses (UTS) for the ropes were obtained by extrapolation from the stress versus logarithm of lifetime graph in Figure 11. Under the UTS the rope would fail immediately, and for the purpose of normalisation 1 second was assumed to be reasonable time that the rope would withstand the ultimate tensile stress before rupturing. Ultimate tensile stresses of 2669, 2670, 2670 and 2681 MPa were obtained for 1.5, 3, 6 and 60 tonne ropes respectively.



(b) Stress in UTS

Fig. 11. – Relationship between applied stress and lifetime of parafil Type G Ropes.

A behaviour similar to that of the bundles of parallel filaments is observed, although the rate of increase in lifetime with respect to decreasing stress is greater here. A decrease of the applied stress from 2380 Mpa to 1700 MPa increases the lifetime from about 7 minutes to 16.3 years as against 7 minutes to 14 years for the bundle of parallel filaments. A further drop of the applied stress to 1423 MPa improves the lifetime dramatically to about 4926 years.

In Figure 12 a comparison of the simulated and empirical results, from the literature, of Parafil Type G ropes [Chambers, 1986; Guimarães, 1988] is presented. The applied stress is expressed as a percentage of the UTS in order to compare with empirical data. As pointed out by G [1988], a reliable prediction is made if the stress is expressed in terms of the UTS, since the size effects on the break load associated with the ropes are at least minimised by this method.

Guimarães' reasoning for normalising the stress-rupture plots in this way was that it removed the bundle theory effect on the short term strength, and also because it fitted the data. But there is a philosophical point that cannot be resolved in this paper. Bundle effects on stress-rupture are taken into account already in the work described here, and there is a very small size effect, as shown in Figure 10. And yet, Guimarães found a considerable size effect on stress-rupture, as shown Figure 12*a*, and it is well known that

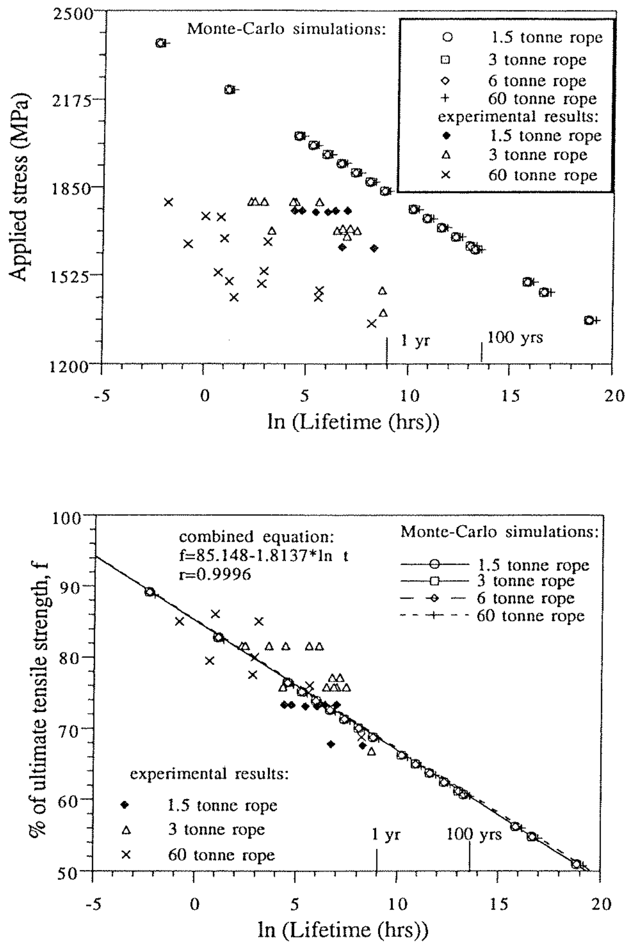


Fig. 12. – Comparaison of Monte-Carlo simulations and experimental results.

there is a bundle size effect on short term strength [A & B, 1995]. It may well be that the solution to this problem lies in the interaction of strength and lifetime variability. The strength of individual components (and its variability) is not taken into account in the present analysis, but it is at least plausible that weak yarns are also the yarns with the shortest lifetime. So ropes that contain a disproportionate number of weak yarns, and so lie at the bottom of the strength spectrum, would also have the shortest lifetimes.

Such a postulate remains pure speculation in the absence of data. As noted already, there is insufficient data on the stress rupture of yarns, and as far as the authors are aware, no data on the interaction with strength effects, either for filaments or yarns.

An extrapolation of the empirical data for Parafil Type G ropes predicts that an applied stress of 50% UTS would cause the rope to fail after 100 years [G, 1988]. The corresponding applied stress predicted from the simulation is 60% UTS. The simulations compare well with the experimental results, despite the assumptions made and the way the ultimate tensile stresses were obtained.

5. Conclusions

Monte-Carlo simulations have been used to study the failure process of bundles of parallel filaments and parallel-lay ropes. This was to give an insight into the process, in view of the fact that there is no satisfactory analytical model to predict creep-rupture.

The method has proved successful in predicting the lifetime, and the variability in the lifetime, of bundles of parallel elements. Good agreement with empirical results was obtained for the maximum stress required for Parafil Type G ropes to survive for over a hundred years when the applied stress was normalized by the ultimate tensile stress.

The following main conclusions are drawn from this study:

(a) There is a lateral size effect associated with the lifetime of bundles of parallel filaments such as yarns. The larger the size of the bundle, the shorter is the lifetime. This behaviour is not observed in parallel-lay ropes (Parafil Type G ropes). Although the lifetimes of Parafil ropes increase slightly with increasing size, the increase is so slight that it can be ignored for practical purposes.

(b) The variability in the lifetime of the fibres has a major influence on the lifetime of the bundle. The larger the scatter, the shorter the time the bundle survives.

(c) The failure of a bundle of parallel elements is precipitated by the rupture of a small proportion of the elements. For a higher applied stress on the bundle, a lesser proportion of failed elements is required to cause the immediate rupture of the bundle.

(d) Under high stresses, bundles of parallel elements made from aramid filaments and yarns fail very quickly, but the lifetime is greatly improved under moderate stresses.

(e) From the simulations, it is shown that under a stress of 60% of the UTS, Parafil Type G ropes should last over 100 years.

(f) There are a number of things that could be done to improve these predictions. Stress rupture data on yarns would, if available, allow the first stage of the analysis, from filaments to yarns to be omitted. Care would be needed that this data was obtained for yarns with the correct amount of twist, since the characteristic length of the yarns is heavily dependent on twist. A study of the scatter of the results would also be useful.

6. Acknowledgements

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