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NONUNIFORM ELASTIC TORSION AND FLEXURE OF MEMBERS WITH ASYMMETRIC CROSS-SECTION

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Abstract—In a previous paper, the authors considered the linear response of a solid prismatic elastic member to a torque which varied along the length of the member. The cross-section considered was only restricted by the requirement of two axes of symmetry. That restriction is removed in the present work, but its removal implies that the response to torque will generally now involve bending, so the treatment has been extended to allow arbitrary loading in torsion/flexure.

It transpired that a satisfactory analysis is incompatible with the concept of a "shear centre", and the existence of a shear centre, as part of a rigorous theory or as an approximation, is clarified in the paper.

NOTATION

A	cross-sectional area
E, G	elastic constants
I_2, I_{23}, I_3, J	second moments of area
L	length of member
M_i	internal couple
P_i	internal force
T	torque ($= M_1$)
e_2	eccentricity of load
i	$= 1, 2, 3$
q_i	linear intensity of distributed load
u_i	displacement
v_i	displacement of centroid of cross-section
w	warping displacement ($= u_1$)
x_i	global Cartesian coordinates
n, t	local coordinates (see Fig. 2 of Ref. [9])

Greek letters

σ_{ij}	stress
ω	torsional rotation

INTRODUCTION

The linear response to load of prismatic elastic members is usually analysed by the superposition of four basic responses, each of which relates a simple deformation component (extension; major-axis bending; minor-axis bending; torsion) to a corresponding stress resultant. When the cross-section does not have two axes of symmetry it is traditional to uncouple these four problems—so that each deformation component is controlled by only one stress resultant—by invoking the concept of a *shear centre*. The axial force stress resultant is taken to act at the centroid of the cross-section, and the zero-extension axes of the bending deformations (the *neutral axes*) also pass through the centroid; the shear force stress resultants, however, are taken to act at the shear centre, defined as that point of a cross-section through which the lines of action of lateral external forces must pass if torsion is to be avoided. Provided the member is appropriately supported the shear centre axis is asserted, by the reciprocal theorem, to be the axis of twist for a pure torsion loading, and the shear centre becomes synonymous with the *centre of twist*, which must further give the axis of a varying rotation when the member is subjected to a *varying* torque, or *nonuniform torsion*.

The centre of twist

The question of the *existence* of a shear centre in a general sense needs clarification. There certainly exists a point in the cross-section at the free end of a cantilever through which a lateral load may be applied without causing torsional rotation (see, for example, Sokolnikoff [1]), and the position of that point is independent of the length of the cantilever. The end cross-section, however, does not remain plane under such a loading; if an extension piece were added, so as to give a cantilever loaded at an intermediate cross-section, the piece would therefore suffer some stress and correspondingly would change the stresses and displacements of the original cantilever, and this change could include twisting. The shear centre for an intermediate load may thus differ from the shear centre for an end load. If so, then we can no longer assert from the reciprocal theorem that the application of a torque anywhere along the member will produce zero lateral displacement of any point on the shear centre axis, for no unique shear centre will exist.

Again, consider the cantilever under *uniform* torsion. If a centre of twist exists and an axial coordinate x_1 has its origin at the fixed end, then the displacement of the *centroid* is given by $a\omega' x_1$, where a is the radius of the centroid from the centre of twist and ω' (or $\mathrm{d}\omega/\mathrm{d}x_1$) is the constant rate of twist. Thus, for small rotations the displaced centroids lie on a straight line. In *nonuniform* torsion, the twist rate ω' varies along the member and the displaced centroids describe a curve. This destroys the intended uncoupling, pure torque loading having caused curvature of the centroidal axis.

There are other problems in the hypothesis that nonuniform pure torque loading will result in pure torsional rotation about an unmoving axis of twist, but the decisive difficulty is that the hypothesis makes it impossible to satisfy the equations of equilibrium. In the torsion analysis that follows we shall postulate a displacement system having four elements: lateral displacements of the centroid $v_2(x_1)$ and $v_3(x_1)$, torsional rotation about the centroid $\omega(x_1)$, and an axial warping displacement $w(x_1, x_2, x_3)$. The concept of a fixed centre of twist will be abandoned, and indeed it becomes the authors' contention that in a general, useful, sense the shear centre does not formally exist. Its role as an approximation will be considered later.

The authors are not of course the first to consider the validity of the shear centre concept. Earlier writers, for example Trefftz [2], Koiter [3], Nowinski [4], Reissner [5] and Pearson [6], studied the position of the shear centre of an arbitrary cross-section and especially the discrepancy arising from two possible definitions of it. Their discussions, however, were mostly restricted to the behaviour of a tip-loaded cantilever. The topic featured also in the search for a general theory of flexure and torsion for closed thin-walled cross-sections, culminating in the work of Hadji-Argyris and Dunne [7], and here a much wider class of problems was considered. The theory was extended in certain respects more recently by Wittrick [8], who showed in passing that in the context of a thin-walled prismatic tube a shear centre, in a general sense, does exist; he gave formulae for its position.

It is not surprising that the convenience of the shear centre has led to its widespread adoption by engineers, not only for the analysis of thin tubes; however, the authors know of no published work establishing its accuracy as an approximation in general. The acceptance of the shear centre may indeed have inhibited the solution of the general torsion/flexure problem, without which the approximation could not be assessed.

NONUNIFORM TORSION AND FLEXURE

The nonuniform torsion of a doubly symmetric elastic member has been treated previously (Burgoyne and Brown [9]), with a solution not restricted to thin-walled cross-sections. Notation from that paper will not be redefined here, but note that the origin of orthogonal coordinates x_i is the *centroid* of an end cross-section, that x_1 is measured axially and that the x_2 axis is not necessarily a principal axis.

The stress resultants are given by

$$P_1 = \int_A \sigma_{11}\,\mathrm{d}A, \tag{1}$$

$$M_1 = \int_A (x_2\sigma_{13} - x_3\sigma_{12})\,dA = T(x_1), \tag{2}$$

$$P_2 = \int_A \sigma_{12}\,dA, \quad P_3 = \int_A \sigma_{13}\,dA \quad \text{(taken at the centroid)}, \tag{3}$$

$$M_2 = \int_A x_3\sigma_{11}\,dA, \quad M_3 = -\int_A x_2\sigma_{11}\,dA. \tag{4}$$

The displacements will be expressed as:

$$\left.\begin{aligned} u_1 &= w(x_1, x_2, x_3) - x_2 v_2'(x_1) - x_3 v_3'(x_1), \\ u_2 &= v_2(x_1) - x_3\omega(x_1), \\ u_3 &= v_3(x_1) + x_2\omega(x_1), \end{aligned}\right\} \tag{5}$$

and Hooke's law, with zero Poisson's ratio, then gives the significant stresses:

$$\sigma_{1j} = \{E(w_{,1} - x_2 v_2'' - x_3 v_3''), G(w_{,2} - x_3\omega'), G(w_{,3} + x_2\omega')\}. \tag{6}$$

If there is no axial force acting ($P_1 = 0$), then Eqn (1) shows, since the origin is at the centroid, that:

$$\int_A w_{,1}\,dA = \int_A (x_2 v_2'' + x_3 v_3'')\,dA = 0. \tag{7}$$

The moment stress resultants become:

$$G\int_A (x_2 w_{,3} - x_3 w_{,2})\,dA + GJ\omega' = G\oint wt\,ds + GJ\omega' = T, \tag{8}$$

where the coordinate t is defined in Fig. 2 of Ref. [9],

$$J = \int_A (x_2^2 + x_3^2)\,dA$$

[cf. Eqn (10) of Ref. [9]), and:

$$\left.\begin{aligned} \int_A x_3 w_{,1}\,dA - I_{23}v_2'' - I_2 v_3'' &= M_2/E, \\ -\int_A x_2 w_{,1}\,dA + I_3 v_2'' + I_{23} v_3'' &= M_3/E, \end{aligned}\right\} \tag{9}$$

where $I_2 = \int_A x_3^2\,dA$, $I_3 = \int_A x_2^2\,dA$, $I_{23} = \int_A x_2 x_3\,dA$. Principal axes are often inconvenient; the labour involved in finding them, so that $I_{23} = 0$, will rarely be repaid.

The only stress-equilibrium equation to be retained, since we wish to establish a theory in terms of stress *resultants*, is the axial one:

$$\sigma_{1j,\,j} = 0,$$

which, by Eqn (6), leads to:

$$G\nabla^2 w + E(w_{,11} - x_2 v_2''' - x_3 v_3''') = 0, \tag{10}$$

where $\nabla^2 \equiv \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$. The corresponding boundary condition is:

$$\frac{\sigma_{i1} x_{i,n}}{G} = w_{,n} + t\omega' = 0 \tag{11}$$

[see Fig. 2 and Eqn (8) of Ref. [9]].

As before, it can readily be shown in view of Eqns (3), (6), (9), (10), (11) and Gauss's theorem that:

$$P_2 = -M_3', \quad P_3 = M_2'.$$

The lateral equilibrium equations:

$$P_2' + q_2 = P_3' + q_3 = 0$$

(where q_2, q_3 are components of distributed load) now combine with Eqns (9) to give:

$$\begin{bmatrix} I_3 & I_{23} \\ I_{23} & I_2 \end{bmatrix} \begin{bmatrix} v_2'''' \\ v_3'''' \end{bmatrix} = \begin{bmatrix} \int_A x_2 w_{,111}\,\mathrm{d}A + \frac{q_2}{E} \\ \int_A x_3 w_{,111}\,\mathrm{d}A + \frac{q_3}{E} \end{bmatrix}. \tag{12}$$

Equations (12) show immediately that nonuniform torsion, leading to a varying warping function w, will in general cause bending of the centroidal axis even in the absence of distributed load (when $q_2 = q_3 = 0$).

The analysis problem for a member under a given loading is to find the displacement functions v_2, v_3, ω and w by solving Eqns (7), (8), (10), (11) and (12) together with appropriate end conditions.

End conditions

Most of the simple end conditions of interest take familiar forms: any of ω, v_2, v_3, v_2' and v_3' may take specified values representing end constraint, or alternatively T, P_2, P_3, M_2 or M_3 may take specified values if the corresponding components of end displacement are unrestrained. It is, however, necessary to consider separately restraint against, or freedom of, warping.

The second and third terms of the expression (5) for u_1, giving a planar displacement of the cross-section, do not involve warping; to achieve zero warping we must have $w = 0$ throughout the cross-section. Similarly, the σ_{11} stresses arising from the bending curvatures v_2'', v_3'' in Eqn (6) provoke no warping, so that the unrestrained warping condition is $w_{,1} = 0$, not $\sigma_{11} = 0$.

Solution by trigonometric series

The choice of series appropriate to a particular problem is governed, as it was in our previous paper, by the end conditions. As a convenient example we shall consider "simple support" at both ends of the member:

$$\left.\begin{aligned} \omega &= 0, \\ v_2 = v_3 &= 0, \\ M_2 = M_3 &= 0, \\ w_{,1} &= 0, \quad \text{for all } x_2,\ x_3. \end{aligned}\right\} \tag{13}$$

In establishing the necessary differentiability of our series we have to consider continuity of the derivatives, and since we may well wish to handle point loads (discontinuous v_2''', v_3''', $w_{,11}$) and torsional couples (discontinuous ω') we shall work from Eqns (8) and (9) rather than from Eqns (12) and an equivalent torsional equation, finding expressions for T, M_2 and M_3 by preliminary equilibrium calculations, if necessary in terms of unknown support reactions.

Then we postulate the following series for the highest-order derivatives needed:

$$\begin{aligned} \omega' &= \sum_{m=1}^{\infty} \frac{m\pi}{L} \Omega_m \cos\frac{m\pi}{L} x_1 + a_0, \\ v_2''' &= -\sum_{m=1}^{\infty} \frac{m^3\pi^3}{L^3} V_{2m} \cos\frac{m\pi}{L} x_1 + b_0, \\ v_3''' &= -\sum_{m=1}^{\infty} \frac{m^3\pi^3}{L^3} V_{3m} \cos\frac{m\pi}{L} x_1 + c_0, \\ w_{,11} &= -\sum_{m=1}^{\infty} \frac{m^2\pi^2}{L^2} W_m(x_2, x_3) \cos\frac{m\pi}{L} x_1 + w_0(x_2, x_3). \end{aligned}$$

Since all lower-order derivatives will be continuous, we can integrate to obtain series expressions for ω, v_2, v_3 and w with polynomial terms added involving arbitrary constants, all of which vanish by virtue of Eqns (13) except for one in the expression for w. The resulting series are differentiable as often as we need:

$$\left.\begin{aligned}\omega &= \sum_{m=1}^{\infty} \Omega_m \sin\frac{m\pi}{L}x_1,\\ v_2 &= \sum_{m=1}^{\infty} V_{2m} \sin\frac{m\pi}{L}x_1,\\ v_3 &= \sum_{m=1}^{\infty} V_{3m} \sin\frac{m\pi}{L}x_1,\\ w &= W_0(x_2, x_3) + \sum_{m=1}^{\infty} W_m(x_2, x_3)\cos\frac{m\pi}{L}x_1.\end{aligned}\right\} \tag{14}$$

When these expressions are inserted into the field equation (10) and its boundary condition (11) we find, in view of the orthogonality of the trigonometric functions:

$$\nabla^2 W_0 = 0 \quad \text{with} \quad W_{0,n} = 0 \text{ on the boundary,} \tag{15}$$

$$\left.\begin{aligned}&\nabla^2 W_m - \frac{m^2\pi^2}{L^2}\frac{E}{G}W_m = -\frac{m^3\pi^3}{L^3}\frac{E}{G}(x_2 V_{2m} + x_3 V_{3m}),\\ &\text{with} \quad W_{m,n} = -\frac{m\pi}{L}t\Omega_m \quad \text{on the boundary}\\ &\text{and} \quad \int_A W_m \, \mathrm{d}A = 0 \quad \text{from Eqn (7).}\end{aligned}\right\} \quad (m = 1, 2, \ldots) \tag{16}$$

We see immediately from Eqn (15) that W_0 is a constant, whose value controls axial rigid-body movement.

The linearity of the system of Eqns (16) suggests a solution in the form:

$$W_m = \alpha_m(x_2, x_3)\Omega_m + \beta_m(x_2, x_3)V_{2m} + \gamma_m(x_2, x_3)V_{3m}, \tag{17}$$

where α_m is defined by the system:

$$\left.\begin{aligned}&\nabla^2\alpha_m - \frac{m^2\pi^2}{L^2}\frac{E}{G}\alpha_m = 0, \quad \int_A \alpha_m \, \mathrm{d}A = 0,\\ &\alpha_{m,n} = -\frac{m\pi}{L}t \quad \text{on the boundary}\end{aligned}\right\} \tag{18}$$

and β_m, γ_m by the systems:

$$\left.\begin{aligned}&\nabla^2\beta_m - \frac{m^2\pi^2}{L^2}\frac{E}{G}\beta_m = -\frac{m^3\pi^3}{L^3}\frac{E}{G}x_2, \quad \int_A \beta_m \, \mathrm{d}A = 0,\\ &\beta_{m,n} = 0 \quad \text{on the boundary,}\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}&\nabla^2\gamma_m - \frac{m^2\pi^2}{L^2}\frac{E}{G}\gamma_m = -\frac{m^3\pi^3}{L^3}\frac{E}{G}x_3, \quad \int_A \gamma_m \, \mathrm{d}A = 0,\\ &\gamma_{m,n} = 0 \quad \text{on the boundary,}\end{aligned}\right\} \tag{20}$$

Each of these three systems is of the form:

$$\nabla^2 f - Cf = X, \tag{a}$$

$$f_{,n} = \lambda t \quad \text{on the boundary,} \tag{b}$$

$$\int_A f \mathrm{d}A = 0. \tag{c}$$

It is readily shown by Gauss's theorem that $\int_A \nabla^2 f \mathrm{d}A = \oint f_{,n} \mathrm{d}s$, where $\oint \ldots \mathrm{d}s$ traverses the whole perimeter of the cross-section, and it follows that any function satisfying (a) and (b) will automatically satisfy:

$$C \int_A f \mathrm{d}A = \lambda \oint t \, \mathrm{d}s - \int_A X \, \mathrm{d}A.$$

If (n_2, n_3) gives the components of the outward normal to the element ds, then $\oint t \, \mathrm{d}s = \oint (x_2 n_3 - x_3 n_2) \mathrm{d}s = 0$ by Gauss's theorem; furthermore $\int_A X \, \mathrm{d}A = 0$ for all three systems, since the origin is at the centroid. As $C \neq 0$, it follows that equation (c) above is automatically satisfied by solutions of (a) and (b).

Systems of equations like Eqns (18), (19) or (20) were met in the previous paper [9]; a numerical solution will usually be necessary. With α_m, β_m and γ_m thus established numerically, the warping function $W_m (m = 1, 2, \ldots)$ will be known once the coefficients Ω_m, V_{2m}, V_{3m} have been found.

Evaluation of the Fourier coefficients

In solving Eqns (8) and (9) for the coefficients we shall need certain integrals of the functions α_m, β_m and γ_m:

$$\alpha_{m1} \equiv \oint \alpha_m t \, \mathrm{d}s = \int_A (x_2 \alpha_{m,3} - x_3 \alpha_{m,2}) \mathrm{d}A,$$

$$\alpha_{m2} \equiv -\int_A x_2 \alpha_m \mathrm{d}A, \quad \alpha_{m3} \equiv -\int_A x_3 \alpha_m \mathrm{d}A,$$

with similar notation for β_m and γ_m.

In view of the orthogonality of the trigonometric functions, the equations give:

$$\begin{bmatrix} \alpha_{m1} + \dfrac{m\pi J}{L} & \beta_{m1} & \gamma_{m1} \\ \alpha_{m2} & \beta_{m2} + \dfrac{m\pi I_3}{L} & \gamma_{m2} + \dfrac{m\pi I_{23}}{L} \\ \alpha_{m3} & \beta_{m3} + \dfrac{m\pi I_{23}}{L} & \gamma_{m3} + \dfrac{m\pi I_2}{L} \end{bmatrix} \begin{bmatrix} \Omega_m \\ V_{2m} \\ V_{3m} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{GL} \displaystyle\int_0^L T \cos \frac{m\pi}{L} x_1 \, \mathrm{d}x_1 \\ -\dfrac{2}{m\pi E} \displaystyle\int_0^L M_3 \sin \frac{m\pi}{L} x_1 \, \mathrm{d}x_1 \\ \dfrac{2}{m\pi E} \displaystyle\int_0^L M_2 \sin \frac{m\pi}{L} x_1 \, \mathrm{d}x_1 \end{bmatrix}. \tag{21}$$

For a given loading in the form $T(x_1)$, $M_2(x_1)$, $M_3(x_1)$, these equations determine the constants $(\Omega_m, V_{2m}, V_{3m})$ to any order of m, and back-substitution into Eqns (14) and (6) gives the displacements and stresses.

This, then, is the general solution for the flexural/torsional response to arbitrary loading of any simply supported uniform elastic member, provided its displacements are small. Other support conditions can be handled similarly, using the series set out in the previous paper [9].

THE SHEAR CENTRE

Suppose that the loading $q(x_1)$ consists entirely of lateral forces in the x_3-direction, and that it is arbitrarily distributed along the member with a uniform eccentricity to the centroid of e_2. Suppose further that the only point loads are the end support reactions, which also have an eccentricity of e_2. Then the existence of a shear centre would imply that a value of e_2

can be found which will result in zero twisting anywhere in the member if one cross-section (say at one end) is held against torsional rotation.

From equilibrium considerations we have $T' = -qe_2$ and $M_2'' = -q$, and with simple-support end conditions we can integrate by parts to find:

$$\int_0^L T\cos\frac{m\pi x_1}{L}\,dx_1 = \frac{Le_2}{m\pi}\int_0^L q\sin\frac{m\pi x_1}{L}\,dx_1,$$

$$\int_0^L M_2\sin\frac{m\pi x_1}{L}\,dx_1 = \frac{L^2}{m^2\pi^2}\int_0^L q\sin\frac{m\pi x_1}{L}\,dx_1,$$

Putting $\int_0^L q\sin(m\pi x_1/L)\,dx_1 = Q_m$ we can write Eqn (21) as:

$$[B]\{\Omega_m, V_{2m}, V_{3m}\} = \left\{\frac{2e_2}{m\pi G}Q_m, 0, \frac{2L^2}{m^3\pi^3 E}Q_m\right\}$$

and if $[\tilde{B}]$ is the inverse of $[B]$, the condition for zero twist ($\Omega_m \equiv 0$) becomes:

$$e_2 = -\frac{L^2}{m^2\pi^2}\frac{G}{E}\frac{\tilde{B}_{13}}{\tilde{B}_{11}}. \tag{22}$$

This, then, is one of the coordinates of the "shear centre", and *for the shear centre to exist as a fixed point for all loading, e_2 must be independent of m.* The matrix $[B]$ of Eqn (21) appears to be a complex function of m (so independence seems unlikely), and the example which follows displays a steady drift of this e_2 position with varying m.

Example

Figure 1 shows a mono-symmetric cross-section which may incontrovertibly be classified as "thin-walled". The symmetry compels $I_{23} = 0$ and also $\alpha_{m2} = \beta_{m1} = \beta_{m3} = \gamma_{m2} = 0$, and the displacement v_2 is uncoupled from ω and v_3, being provoked only by loading in the x_2-direction.

The computer program of our previous paper [9] was adapted to evaluate the field functions α_m and γ_m at the points of a reasonably fine mesh (17 nodes spaced unequally across the thickness of the flange, for example, and 71 along its length). There were convergence problems, arising it seemed from the "floating" nature of the boundary condition, which specifies not the magnitude of the field function but the values of its normal derivative. The difficulty was handled by the technique described in the Appendix.

For distributed loading restricted to the x_3-direction, Eqn (21) now reduces to:

$$\begin{bmatrix} \alpha_{m1} + \dfrac{m\pi J}{L} & \gamma_{m1} \\ \alpha_{m3} & \gamma_{m3} + \dfrac{m\pi I_2}{L} \end{bmatrix} \begin{bmatrix} \Omega_m \\ V_{3m} \end{bmatrix} = \begin{bmatrix} \dfrac{2e_2}{m\pi G}Q_m \\ \dfrac{2L^2}{m^3\pi^3 E}Q_m \end{bmatrix} \tag{23}$$

and Eqn (22) becomes:

$$e_2 = \frac{L^2}{m^2\pi^2}\frac{G}{E}\frac{\gamma_{m1}}{\gamma_{m3} + \dfrac{m\pi I_2}{L}}. \tag{24}$$

Traditional theory places a shear centre at a distance $a = 0.3033d$ outside the cross-section, as shown in Fig. 1. With $L/d = 20$, the corresponding quantities a_m derived from Eqn (24) for different values of m are listed in Table A1, together with the integrals α_{m1}, etc., and the outcome is illustrated in Fig. 2. There is a clear drift of the "shear centre" with increasing m, which implies that *for a sinusoidally distributed load $q = Q\sin m\pi x_1/L$ to cause no torsional rotation, it must be applied at a position which depends upon the value of m.*

It is now possible, for this particular beam, to examine the inaccuracy involved in adopting the shear centre as an approximation. Suppose this same simply supported beam to carry a uniformly distributed load q, both the load and the support reactions passing

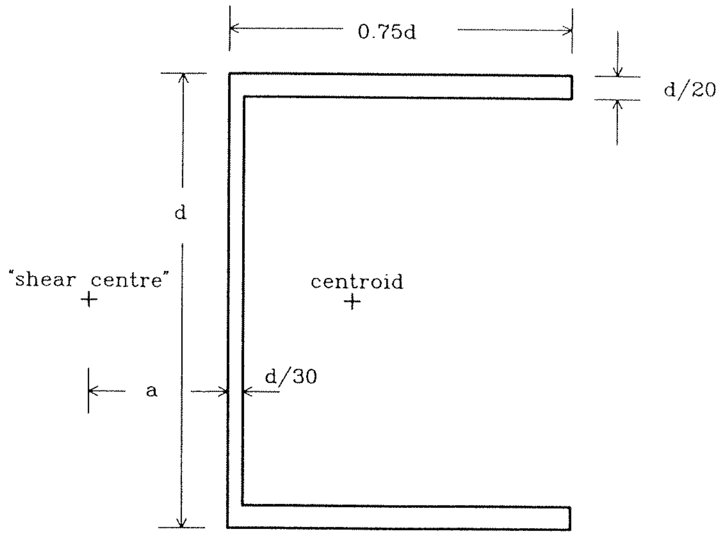

FIG. 1. A thin-walled cross-section.

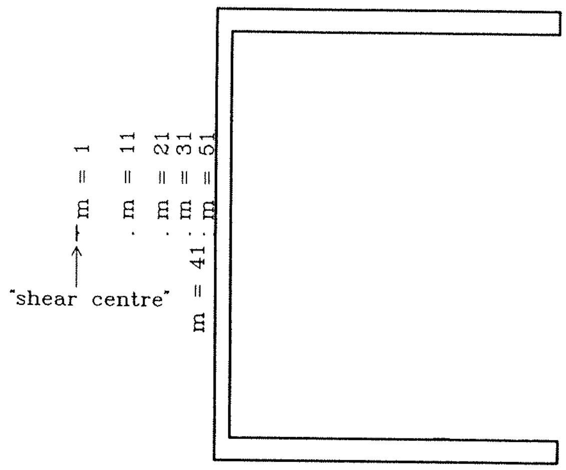

FIG. 2. Position of a distributed load $q = Q\sin(m\pi x_1/L)$ to give no torsional rotation.

through the traditional shear centre. We shall compute the maximum torsional rotation which results.

The accurate traditional value of e_2 is $-0.57541602d$, and for this loading:

$$Q_m = 2qL/m\pi \quad (m = 1, 3, 5 \ldots),$$
$$= 0 \quad (m = 2, 4, 6, \ldots).$$

If B_m denotes the determinant of the first matrix of Eqn (23), the values Ω_m are:

$$\Omega_m = \frac{4qL}{m^2\pi^2 B_m}\left[\left(\gamma_{m3} + m\pi\frac{I_2}{L}\right)\frac{e_2}{G} - \frac{\gamma_{m1}L^2}{m^2\pi^2 E}\right] \quad (m = 1, 3, 5 \ldots),$$
$$= 0 \qquad (m = 2, 4, 6, \ldots).$$

The maximum bending stress will be $qL^2d/16I_2$, and we shall limit q to such a value as will make this one half of the ultimate tensile stress of the material (listed in the following table as σ). Then for a given material the mid-span rotation ω is found from Eqn (14), in which all values are now known numerically. The authors are indebted to Mr P. C. Gasson, of the Department of Aeronautical Engineering, Imperial College, for guidance on typical material values, which lead to the results shown in Table 1. Other loading cases and other

TABLE 1. RESULTS FOR TYPICAL MATERIAL VALUES

Material	Material values used $E(\mathrm{kN\,mm^{-2}})$	$\sigma(\mathrm{N\,mm^{-2}})$	ω_{max} (degrees)
Mild steel	207	200	0.12
Aluminium alloy	72	250	0.4
Titanium alloy	115	550	0.6
H.S. steel	207	1100	0.6
Carbon fibre/epoxy composite	101	710	0.9
Glass fibre composite	69	1000	1.7

cross-sections may of course give higher figures, but these results do suggest that significant unexpected torsional rotations are most likely to occur in the use of modern high-strength materials.

CONCLUSIONS

For beams of asymmetric cross-section it is apparent that a varying torque will cause bending as well as twisting, so there is no worthwhile simplification in considering torsional loading alone. The analysis here has treated a general loading of applied bending moments and torque varying continuously or discontinuously along the member. The equations to be solved are Eqns (7), (8), (10), (11) and (12), and a solution by trigonometric series involves computing three warping functions for the cross-section (α, β, γ), the derivations of which fortunately all take the same basic form as was encountered in the previous paper [9]. They are independent of the loading.

For a given cross-section, the three functions and certain integrals of them can be found numerically, and the integrals represent the essential geometric properties of the beam, analogous to second moments of area and torsion constants in ordinary beam theory. Thus, by a systematic if slightly cumbersome process, the response of a beam of arbitrary cross-section to arbitrary loading can at last be given an engineering analysis.

A nontrivial example of such a solution for a symmetric cross-section was given in the previous paper, and instead of demonstrating this practical feasibility again here the authors have used the general theory to elucidate the status of the traditional "shear centre". The conclusions are as follows:

(1) In a general, useful, sense the shear centre does not formally exist. There is no single point in a general cross-section through which lateral loads anywhere along the beam can be applied without causing twisting.

(2) A shear centre found by and used in a traditional analysis of thin-walled cross-sections may result in a fair approximation if the load is uniformly distributed, but the error involved will be more pronounced with modern high-strength materials.

REFERENCES

1. I. S. SOKOLNIKOFF, *Mathematical Theory of Elasticity*. 2nd Edn, pp. 244–248. McGraw-Hill, New York and London (1956).
2. E. TREFFTZ, Uber den Schubmittelpunkt in einem durch eine Einzellast gebogenen Balken. *ZAMM* **15**, 220–225 (1935).
3. W. T. KOITER, The torsional center of beams loaded by shearing forces. *A.R.C. Report* 4283, Strut 482 (1939).
4. J. NOWINSKI, Flexure of beams by terminal loads. *Arch. Mech. Stos* **2**, 89–105 (1950).
5. E. REISSNER, On torsion with variable twist. *Ost. Ing.-Arch.* **9**, 218–224 (1955).
6. C. E. PEARSON, Remarks on the center of shear. *ZAMM* **36**, 94–96 (1956).
7. J. HADJI-ARGYRIS and P. C. DUNNE, General theory of cylindrical and conical tubes under torsion and bending loads. *J. R. Aero. Soc.* **51**, 199–269 (1947).
8. W. H. WITTRICK, Analysis of shear lag in cylindrical tubes of arbitrary cross-section. *Proc. R. Soc. Lond.* **A389**, 259–277 (1983).
9. C. J. BURGOYNE and E. H. BROWN, Non-uniform elastic torsion. *Int. J. Mech. Sci.* **36**, 23–38 (1994).

APPENDIX

The coefficients α_m, β_m, γ_m of Eqn (17) are field functions of the cross-section, governed by the system:

$$\nabla^2 f - Cf = X,$$

$$f_{,n} \text{ given on the boundary,}$$

where C is a known constant and X a known field function linear in x_2 or x_3. The computer program solved the corresponding finite difference formulation for a rectangular mesh of variable spacing, where the ratio of adjacent mesh-lengths never exceeded 2:1. This reduced the total number of nodes by concentrating them in areas where the function might undergo rapid variation. Nonetheless, half the cross-section of Fig. 1 was represented by 1687 nodes.

The large number of simultaneous equations was solved by the method of Successive Over-Relaxation, but convergence was never fast and a solution accurate to six significant figures could involve 10,000 iterations. Symmetry of this cross-section compelled $\alpha_m = \gamma_m = 0$ on the centre-line [β_m was not required for this loading; on the centre-line ($x_3 = 0$) it would have $\beta_{m,3} = 0$], and convergence was much improved if the function was held to an arbitrary fixed value $f = f_1$ (nonzero) at one other node well away from the centre-line. For this to be possible, the boundary constraint on $f_{,n}$ had to be relaxed at one node. If the arbitrary fixed value were then changed to $f = f_2$, giving a second "solution", the two "solutions" could finally be superimposed so as to satisfy the neglected constraint on $f_{,n}$. This device did significantly improve performance, even at the cost of solving the equations twice.

TABLE A1. PROPERTIES OF THE CROSS-SECTION SHOWN IN FIG. 1

m	α_{m1}/d^3	α_{m3}/d^4	γ_{m1}/d^2	γ_{m3}/d^3	a_m/d
1	− 0.00388265	− 0.00163060	− 0.000104550	− 0.000134707	0.3004
3	− 0.01044054	− 0.00353431	− 0.002039460	− 0.002660149	0.2902
5	− 0.01532661	− 0.00374746	− 0.006006551	− 0.008014925	0.2719
7	− 0.01928706	− 0.00334229	− 0.010499336	− 0.014441164	0.2485
9	− 0.02271863	− 0.00284914	− 0.014794151	− 0.021065678	0.2232
11	− 0.02577303	− 0.00241409	− 0.018723932	− 0.027642348	0.1983
13	− 0.02853513	− 0.00205950	− 0.022308648	− 0.034125576	0.1750
15	− 0.03107203	− 0.00177627	− 0.025614426	− 0.040520996	0.1541
17	− 0.03343978	− 0.00155006	− 0.028708476	− 0.046843880	0.1357
19	− 0.03568320	− 0.00136798	− 0.031646398	− 0.053108746	0.1195
21	− 0.03783636	− 0.00121982	− 0.034470446	− 0.059327344	0.1053
23	− 0.03992419	− 0.00109782	− 0.037211308	− 0.065508798	0.0929
25	− 0.04196444	− 0.00099615	− 0.039890660	− 0.071660130	0.0820
27	− 0.04396946	− 0.00091044	− 0.042523514	− 0.077786766	0.0723
29	− 0.04594771	− 0.00083742	− 0.045120104	− 0.083892946	0.0636
31	− 0.04790491	− 0.00077458	− 0.047687292	− 0.089982016	0.0558
33	− 0.04984486	− 0.00072000	− 0.050229602	− 0.096056646	0.0487
35	− 0.05177004	− 0.00067221	− 0.052749932	− 0.102118986	0.0421
37	− 0.05368201	− 0.00063003	− 0.055250072	− 0.108170790	0.0360
39	− 0.05558171	− 0.00059256	− 0.057731044	− 0.114213498	0.0304
41	− 0.05746967	− 0.00055904	− 0.060193358	− 0.120248304	0.0250
43	− 0.05934613	− 0.00052890	− 0.062637170	− 0.126276202	0.0200
45	− 0.06121114	− 0.00050164	− 0.065062402	− 0.132298034	0.0152
47	− 0.06306462	− 0.00047688	− 0.067468822	− 0.138314506	0.0106
49	− 0.06490642	− 0.00045429	− 0.069856096	− 0.144326228	0.0063
51	− 0.06673632	− 0.00043359	− 0.072223830	− 0.150333716	0.0020