Int. J. Mech. Sci. Vol. 36, No. 1, pp. 23–38, 1994
Printed in Great Britain.

0020–7403/94 $6.00 + .00
Pergamon Press Ltd

NONUNIFORM ELASTIC TORSION

C. J. Burgoyne* and E. H. Brown†

*Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ; and †Civil Engineering Department, Imperial College of Science, Technology and Medicine, London SW7 2BU, U.K.

(*Received* 10 *September* 1992; *and in revised form* 12 *May* 1993)

Abstract—Linear and nonlinear uniform torsion of prismatic members has been much studied, but even in the linear case nonuniform torques have been comparatively neglected. Thin-walled cross-sections have received useful approximate treatments by Goodier and others, but a general theory seems still to be lacking.

The linear theory given here corresponds to standard theories for varying elastic flexure. The torque may vary continuously or discontinuouslv along the member but, as with flexure, we deliberately avoid specifying in detail how it is applied.

The present analysis is for cross-sections with two axes of symmetry. The examples given include an important one illustrating the possible sensitivity of beams to eccentricity of loading.

NOTATION

A	cross-sectional area
E, G	elastic constants
J	polar second moment of area
K	Saint-Venant's torsion constant
L	length of member
M_i	internal couple
P_i	internal force
T	torque $(= M_1)$
e	eccentricity of load
i	$= 1, 2, 3$
q	linear intensity of distributed load
u_i	displacement
w	warping displacement $(= u_1)$
x_i	global Cartesian coordinates
$\bar{x}_i, n, t, \phi$	local coordinates (see Fig. 2)

Greek letters

ε_{ij}	strain
σ_{ij}	stress
ν	Poisson's ratio
ω	torsional rotation

INTRODUCTION

Saint-Venant's theory for uniform linear elastic torsion asserts that a prismatic bar under constant torque will undergo linearly varying axial rotation, with warping displacements which will vary over the cross-section but will be constant along the length of the bar. There will be no direct stresses. The solution is exact if the displacement gradients are small, the end torques are applied in a prescribed way and the end cross-sections are free to warp.

Larger rotations have been studied by Weber [1], Cullimore [2], Ashwell [3], and Gregory [4]; small-rotation small-torque behaviour in the presence of large axial forces or bending moments has been considered by Buckley [5], Wagner [6], Biot [7] and Goodier [8]. All these authors treat only thin-walled sections, and their work is brought together and compared in a more general treatment by Gregory [9].

If the ends of the bar are not free to warp, or the loading is not confined to the ends, we have a problem of nonuniform torsion (see for example Fig. 1; with linear elasticity the torsion and bending effects are uncoupled and can be superimposed). Even for small rotations, nonzero direct stresses now result. Approximate theory for thin-walled sections has been given by Timoshenko [10], Goodier [11], Bleich [12], Vlasov [13] and others, but

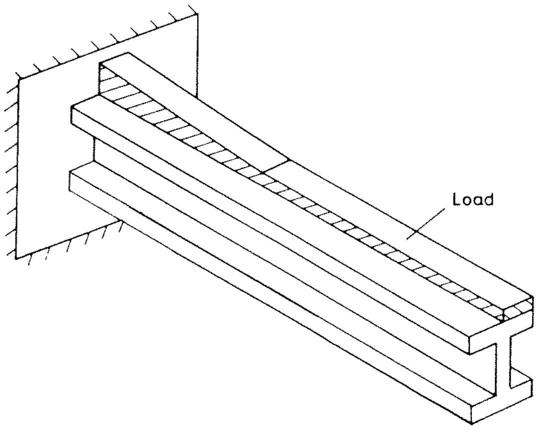

FIG. 1. A cantilever subject to nonuniform torque.

attempts at a more general analysis have been few. Reissner [14] suggested some simplifying approximations, but gave no applications. Sokolnikoff, in the second edition of his well-known book [15], solved the particular problem of a cantilever under linearly varying torque. McMinn [16], in a remarkable paper, attempted a complete solution using three-dimensional elasticity, but unfortunately there appear to be difficulties in expressing any but trivial boundary conditions in the form required, and the numerical computations involved are formidable.

The linear solution given here is in the spirit of other technical beam theories. It considers an elastic isotropic prismatic bar of doubly symmetric, but otherwise arbitrary, solid cross-section,* twisted through small angles by externally applied axial couples which may vary, continuously or discontinuously, along the length of the bar. The end cross-sections may be constrained against axial warping, and we deliberately avoid specifying in detail how the loading is applied to the member, in keeping with all generally useful theories for elastic bars, both torsional and flexural: only the stress *resultants* are prescribed completely, and we seek an approximate analysis of widely applicable generality.

BASIC EQUATIONS

We retain the Saint-Venant concept of individual cross-sections rotating undeformed through a small angle ω about the centroidal axis, which we take as the x_1-coordinate axis, postulating that the member is supported appropriately. In the plane of an end section we take the two axes of symmetry to complete a right-handed Cartesian set, defining coordinates $x_i (i = 1, 2, 3)$. The corresponding stress components will be labelled σ_{ij}.

With $\int_A \ldots \mathrm{d}A$ denoting integration over the whole cross-section, the six stress resultants are specified as follows:

$$P_1 \equiv \int_A \sigma_{11}\,\mathrm{d}A = 0, \tag{1}$$

$$P_2 \equiv \int_A \sigma_{12}\,\mathrm{d}A = 0, \qquad P_3 \equiv \int_A \sigma_{13}\,\mathrm{d}A = 0, \tag{2}$$

$$M_2 \equiv \int_A x_3\,\sigma_{11}\,\mathrm{d}A = 0, \quad M_3 \equiv -\int_A x_2\,\sigma_{11}\,\mathrm{d}A = 0, \tag{3}$$

$$M_1 \equiv \int_A (x_2\,\sigma_{13} - x_3\,\sigma_{12})\mathrm{d}A = T(x_1), \tag{4}$$

where $T(x_1)$ is the variable torque.

* The more complex case of asymmetric cross-sections will be treated in a companion paper, together with the problem of the existence of a *shear centre*.

The displacements will be:

$$\left.\begin{aligned} u_1 &= w(x_1, x_2, x_3), \quad (\text{"warping displacement"}) \\ u_2 &= -\, x_3\omega(x_1), \\ u_3 &= x_2\omega(x_1), \end{aligned}\right\} \tag{5}$$

and from these we can derive components of strain from the relation:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

where the comma abbreviation denotes derivatives: $u_{i,j} \equiv \partial u_i/\partial x_j$ etc.

Isotropic elasticity, with Young's modulus E, shear modulus G and Poisson's ratio ν $[G = E/2(1 + \nu)]$, implies stresses given by:

$$\sigma_{ij} = \begin{bmatrix} \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)} w_{,1} & G(w_{,2} - x_3\omega') & G(w_{,3} + x_2\omega') \\ G(w_{,2} - x_3\omega') & \dfrac{\nu E}{(1+\nu)(1-2\nu)} w_{,1} & 0 \\ G(w_{,3} + x_2\omega') & 0 & \dfrac{\nu E}{(1+\nu)(1-2\nu)} w_{,1} \end{bmatrix},$$

where ω' stands for $\mathrm{d}\omega/\mathrm{d}x_1$. If Poisson's ratio is zero we have $\sigma_{22} = \sigma_{33} = 0$, and the stresses then become:

$$\sigma_{ij} = \begin{bmatrix} Ew_{,1} & G(w_{,2} - x_3\omega') & G(w_{,3} + x_2\omega') \\ G(w_{,2} - x_3\omega') & 0 & 0 \\ G(w_{,3} + x_2\omega') & 0 & 0 \end{bmatrix}. \tag{6}$$

We shall still accept these expressions where $\nu \neq 0$, as our essential approximation. Similar disregard of small values σ_{22} and σ_{33} is familiar in the technical theory of the flexure of beams.

We cannot now expect to satisfy differential equations of equilibrium in the transverse directions, and indeed we do not wish to do so, for we mean to avoid specifying in detail how the varying torque $T(x_1)$ is applied; standard bending theory considers a bending moment $M(x_1)$ in the same way.

In the more important axial direction (more important because of our concern with variable axial warping), equilibrium requires:

$$\sigma_{1j,j} = 0$$

(with summation implied over repeated suffixes), or [by Eqn (6)]:

$$\nabla^2 w + \frac{E}{G} w_{,11} = 0, \tag{7}$$

where $\nabla^2 \equiv \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$. The corresponding axial-direction boundary condition asserts zero axial surface shear stress over the whole length of the member. Figure 2 shows a cross-section with local coordinates $\bar{x}_i$ parallel and normal to a particular boundary element; the $\bar{x}_1$-axis coincides with the x_1-axis. Then we have $\bar{\sigma}_{31} = 0$, where $\bar{\sigma}_{ij}$ are the stress components in the $\bar{x}_i$ directions. If c_{ij} represents the direction cosine $\cos(x_i, \bar{x}_j)$ we have, since $c_{j1} = \delta_{j1}$ (Kronecker delta):

$$\bar{\sigma}_{31} = c_{i3}\, c_{j1}\, \sigma_{ij} = c_{i3}\, \sigma_{i1} = 0$$

on the boundary.

Furthermore, $c_{i3} = \partial x_i/\partial \bar{x}_3 = \{0, \sin\phi, \cos\phi\}$, so that the stress values from Eqn (6) give:

$$\frac{1}{G} c_{i3}\, \sigma_{i1} = w_{,2} \frac{\partial x_2}{\partial \bar{x}_3} + w_{,3} \frac{\partial x_3}{\partial \bar{x}_3} - \omega'(x_3 \sin\phi - x_2 \cos\phi) = 0.$$

With $(\bar{x}_2, \bar{x}_3)$ for each boundary element written as (t, n), the boundary condition thus takes

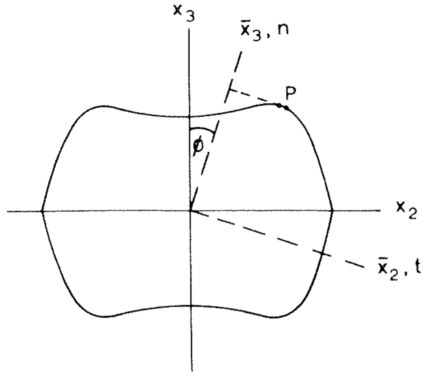

FIG. 2. The $\bar{x}_3$ axis takes the direction of the exterior normal at P.

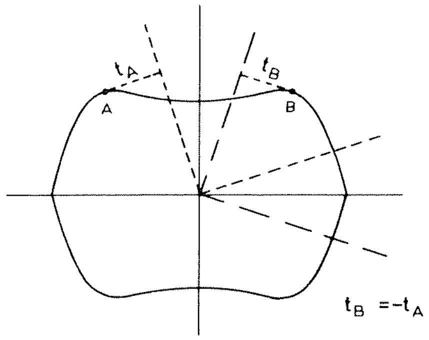

FIG. 3. Anti-symmetry of the coordinate t.

the final form:

$$\frac{\partial w}{\partial n} = -\, t\omega'. \tag{8}$$

THE STRESS RESULTANTS

The stress expressions in Eqn (6) are now inserted in conditions (1)–(4). Equation (1) becomes:

$$\int_A w_{,1}\, \mathrm{d}A = 0. \tag{9}$$

Symmetry about the x_3-axis gives sign reversal of t (Fig. 3), and compels anti-symmetry of w in view of Eqns (7) and (8), except for a term $c_0 + c_1 x_1$, where c_0 and c_1 are constants. Equation (9) therefore only requires $c_1 = 0$, and to avoid rigid-body movement we take $c_0 = 0$ also.

With the polar second moment of area $\int_A (x_2^2 + x_3^2)\,\mathrm{d}A$ denoted by J, Eqn (4) leads to:

$$T(x_1) = GJ\omega' + G\int_A \{(wx_2)_{,3} - (wx_3)_{,2}\}\, \mathrm{d}A$$

$$= GJ\omega' + G\oint wt\, \mathrm{d}s, \tag{10}$$

by Gauss's theorem, where $\oint \ldots \mathrm{d}s$ denotes integration around the boundary of the cross-section. This form for variable T contrasts with Saint-Venant's form, but must of course become identical with it if T is a constant.

The second of Eqns (2) gives, by Eqn (7):

$$P_3 = G\int_A (w_{,3} + x_2\omega')\,\mathrm{d}A$$

$$= G\int_A \{x_3(w_{,3} + x_2\omega')\}_{,3}\,\mathrm{d}A + G\int_A \{x_3(w_{,2} - x_3\omega')\}_{,2}\,\mathrm{d}A + E\int_A x_3 w_{,11}\,\mathrm{d}A$$

$$= G\oint x_3(w_{,n} + t\omega')\,\mathrm{d}s + \int_A x_3\sigma_{11,1}\,\mathrm{d}A,$$

or:

$$P_3 = M'_2,$$

by Eqns (3) and (8), where $M' \equiv \mathrm{d}M/\mathrm{d}x_1$. Similarly:

$$P_2 = -M'_3.$$

It follows that if we can satisfy Eqns (3) for all x_1 then Eqns (2) will be satisfied automatically.

Equations (3), with the stresses of Eqn (6), become:

$$M_2 = E\int_A x_3 w_{,1}\,\mathrm{d}A = 0, \quad M_3 = -E\int_A x_2 w_{,1}\,\mathrm{d}A = 0,$$

and both are satisfied by the anti-symmetry of w for all values of x_1.

We are left therefore to solve Eqns (7), (8) and (10), with an anti-symmetric w-distribution which will give $w = 0$ on the axes of symmetry. We first distinguish two problems:

The first torsion problem. Given a prismatic member, twisted through an arbitrary but *specified* angle $\omega(x_1)$, find the warping function $w(x_1, x_2, x_3)$ and the corresponding variation of torque $T(x_1)$. It is implicit that any variation $T(x_1)$ can be impressed upon the member by a suitable distribution of externally applied forces.

The second torsion problem. Given a prismatic member, its support conditions and an arbitrary but specified torque $T(x_1)$ to which it is subjected, find the warping function $w(x_1, x_2, x_3)$ and the corresponding twist $\omega(x_1)$. The torque may be statically indeterminate and so not wholly specified, in which event there will be a compatibility condition such as:

$$\omega(L) - \omega(0) = 0. \tag{11}$$

END CONDITIONS

There are two alternative end conditions commonly specified with respect to warping:

$$\left.\begin{array}{l}\text{(a) warping unrestrained } (\sigma_{11} = 0)\\ \text{(b) warping wholly restrained } (w = 0)\end{array}\right\}\ \text{for all } x_2, x_3.$$

No single end condition on the variable ω will ensure the satisfaction of either of these, but instead we can derive an infinite sequence of conditions on ω, each of which independently is a necessary condition; only the full sequence yields a *sufficient* condition.

In the following discussion, bracketed superscripts will denote differentiations with respect to x_1 [$w^{(3)} \equiv w_{,111}$, etc.], and a subscript zero will indicate the value of the quantity at an end of the member. Thus w_0, for example, will be a function of x_2 and x_3, but not of x_1.

End warping unrestrained

This requires:

$$\sigma_{110} = Ew^{(1)}{}_0 = 0 \quad \text{for all } x_2, x_3.$$

The boundary condition (8), true for all x_1, can be differentiated to give:

$$w^{(1)}_{,n} = -t\omega'',$$

on the boundary of the cross-section.

Now, if $w^{(1)}{}_0$ is to be zero for all x_2 and x_3, then $w^{(1)}_{,n0}$ must be zero all around the perimeter, and since t is not everywhere zero (except for circular sections, which do not warp at all) it follows that $\omega''{}_0 = 0$. This is a necessary (but not sufficient) condition for $w^{(1)}{}_0$ to vanish.

We can extend this result by differentiating the governing differential equation (7) as many times as we wish:

$$w^{(3)} = -\frac{G}{E}\nabla^2 w^{(1)},$$

$$w^{(5)} = -\frac{G}{E}\nabla^2 w^{(3)} = \left(-\frac{G}{E}\right)^2 \nabla^4 w^{(1)},$$

$$w^{(r)} = \left(-\frac{G}{E}\right)^{\left(\frac{r-1}{2}\right)} \nabla^{r-1} w^{(1)} \quad \text{(for } r \text{ odd and } \geqslant 3).$$

These equations apply throughout the member, but if we now apply the end condition $w^{(1)}{}_0 = 0$ for all x_2, x_3, it follows that:

$$w^{(3)}{}_0 = w^{(5)}{}_0 = w^{(7)}{}_0 = \ldots\ldots\ldots = 0, \quad \text{for all } (x_2, x_3)$$

and:

$$w^{(3)}_{,n0} = w^{(5)}_{,n0} = w^{(7)}_{,n0} = \ldots. = 0,$$

on the perimeter.

We can differentiate Eqn (8) with respect to x_1 as often as we wish to get:

$$w^{(r)}_{,n} = -t\omega^{(r+1)}, \tag{12}$$

on the perimeter, and the final end conditions become:

$$\omega^{(2)}{}_0 = \omega^{(4)}{}_0 = \omega^{(6)}{}_0 = \ldots = 0. \tag{13}$$

End warping wholly restrained

Again we can make use of the weaker condition that $w_{,n0} = 0$ on the perimeter. Substitution of this result into the boundary condition (8) gives $\omega^{(1)}{}_0 = 0$ directly.

We can also differentiate Eqn (7) with respect to x_1, as before:

$$w^{(r)} = \left(-\frac{G}{E}\right)^{r/2} \nabla^r w, \quad \text{for } r \text{ even and } \geqslant 2.$$

If $w_0 = 0$ for all x_2 and x_3, it follows that:

$$w^{(2)}{}_0 = w^{(4)}{}_0 = w^{(6)}{}_0 = \ldots = 0, \quad \text{for all } x_2, x_3,$$

$$w^{(2)}_{,n0} = w^{(4)}_{,n0} = w^{(6)}_{,n0} = \ldots = 0$$

on the perimeter, and this can be substituted into the differentiated boundary condition (12) to give:

$$\omega^{(1)}{}_0 = \omega^{(3)}{}_0 = \omega^{(5)}{}_0 = \ldots = 0. \tag{14}$$

The anomaly of the torque

We have just seen that at an end where warping is restrained ($w = 0$ or all x_2, x_3) the derivative ω' vanishes, and in this event Eqn (10) clearly cannot locally be satisfied. The anomaly results directly from the original concept of cross-sections rotating undeformed. Unless a singularity occurs where the surface of the member cuts its end plane, the boundary condition $\bar{\sigma}_{31} = 0$ [Eqn (8)] will compel $\omega'_0 = 0$ for all noncircular sections, and the stress expressions (6) will give $\sigma_{12} = \sigma_{13} = 0$. Equation (4) cannot then be satisfied.

We must conclude that undeformed rotating cross-sections near a restrained end are impossible, and the true displacement field is complex. Solutions based on the boundary conditions (14) appear to give sensible results for the member as a whole, however, and of course $\omega'_0 = 0$ has been successfully used as a boundary condition in many analyses which treat warping restraint of thin-walled sections.

METHODS OF SOLUTION

The nonuniform torsion problem has now been reduced to the solution of the following equations:

$$\nabla^2 w + \frac{E}{G} w_{,11} = 0, \tag{7}$$

$$\frac{\partial w}{\partial n} = -t\omega' \quad \text{on the boundary of the cross-section, for all } x_1, \tag{8}$$

$$T(x_1) = GJ\,\omega' + G \oint wt\,\mathrm{d}s, \tag{10}$$

$$w_{,1} = 0 \text{ or } \omega^{(r)} = 0 \quad \text{at an unrestrained end of a member, where } r = 2, 4, 6, \ldots. \tag{13}$$

$$w = 0 \text{ or } \omega^{(r)} = 0 \quad \text{at a restrained end of a member, where } r = 1, 3, 5, \ldots. \tag{14}$$

w is required to take a zero value on the axes of symmetry.

Two approaches to the solution of these equations have been explored. The first was prompted by analogy with earlier thin-wall approximate analyses which, in place of Eqn (10), have proposed an equation $T = GK\omega' - E\Gamma\omega'''$, where K and Γ were cross-sectional properties. With this in mind a solution was sought in the form:

$$w(x_1, x_2, x_3) = \sum_{r=0}^{\infty} g_r(x_2, x_3)\,\omega^{(r)}(x_1). \tag{15}$$

The twist $\omega(x_1)$ here will obviously be related to the varying torque $T(x_1)$, but the coefficients g_r, independent of x_1, will be the same *for all values of the torque.*

The first torsion problem

Substituting the form (15) into the earlier equations we readily find, on equating the coefficients of $\omega^{(0)}$ to zero:

$$\nabla^2 g_0 = 0, \quad \text{with} \quad \frac{\partial g_0}{\partial n} = 0 \quad \text{on the boundary,}$$

giving g_0 to be a constant, which must be zero since Eqn (9) leads to:

$$\int_A g_r\,\mathrm{d}A = 0 \quad (r = 0, 1, 2, \ldots), \tag{16}$$

if the coefficient of each derivative $\omega^{(r+1)}$ is to vanish.

A similar argument, for successive even values of r, gives:

$$g_r = 0 \quad (r \text{ even or zero}).$$

To find the values of g_r for r odd requires the solution of a sequence of Laplace or Poisson equations with Neumann-type boundary conditions:

$$\left.\begin{aligned} &\nabla^2 g_1 = 0 \text{ with } g_{1,n} = -t \text{ on the boundary,} \\ &\nabla^2 g_r = -\frac{E}{G} g_{r-2} \text{ with } g_{r,n} = 0 \text{ on the boundary } (r \geqslant 3). \end{aligned}\right\} \tag{17}$$

This system can be solved (numerically if necessary) to give as many of the g_r functions as are needed for convergence, and the required warping function is then given by Eqn (15). We turn finally to Eqn (10) to find the torque:

$$T(x_1) = G\{K_1\omega' + K_3\omega^{(3)} + K_5\omega^{(5)} + \ldots\}, \tag{18}$$

where:

$$K_1 = J + \oint g_1 \, t \, \mathrm{d}s,$$

$$K_r = \oint g_r \, t \, \mathrm{d}s \quad (r \text{ odd}, \geqslant 3).$$

The stiffnesses GK_1 and GK_3 can be shown to coincide with GK and $-E\Gamma$ of the earlier theories in the restricted cases where the latter are applicable (Burgoyne[17]). Thus, K_1 is the Saint-Venant torsion constant.

The second torsion problem

The solutions of the first problem give the torque $T(x_1)$ and the warping function $w(x_1, x_2, x_3)$ for arbitrary continuous rotation functions $\omega(x_1)$. Solutions for the rotation in the second problem are members of this class, so Eqn (18) is still valid, now relating an unknown twist ω to a specified torque $T(x_1)$ in terms of the known, constant, cross-sectional properties K_r.

The procedure must be to truncate the series of Eqn (18), and solve the resulting linear differential equation under an appropriate number of the end conditions (13) or (14). The method succeeds or fails according to whether the result does or does not converge with an increasing number of terms retained in Eqn (18). The authors are indebted to Professor F. G. Leppington,* whose investigation of some of the convergence problems involved leads us to expect success in some but not all cases.

SOLUTION BY TRIGONOMETRIC SERIES

Our second approach appears more reliable, but it lacks the advantages of simple stiffness expressions to be compared with existing thin-wall theory and of the reduction of the second torsion problem to an ordinary differential equation with end conditions which can be inserted at a late stage in the analysis. Our choice of trigonometric series will depend upon the particular end conditions to be satisfied, and we shall treat three separate cases:

Case A—warping unrestrained at $x_1 = 0$ and $x_1 = L$;
Case B—warping prevented at $x_1 = 0$, unrestrained at $x_1 = L$; and
Case C—warping prevented at $x_1 = 0$ and $x_1 = L$.

In addition to these warping conditions we shall consider the rotation ω to be zero at one end, and either the torque or the rotation to take a specified value at the other. Case A requires a further restriction to control axial rigid-body movement; in Eqn (23) below we shall put the mean warping displacement equal to zero $[\int_A w \, \mathrm{d}A = 0$, which will hold for all x_1 in view of Eqn (9)].

We first expand $w_{,11}$ [needed for Eqn (7)] as an appropriate series and integrate twice to obtain series for $w_{,1}$ and w, with constants of integration; the series is "appropriate" if these latter vanish in satisfying the end conditions. The resulting series can thus be legitimately differentiated twice term by term:

$$\left.\begin{aligned} w &= W_0 + \sum_{m=1}^{\infty} W_m \cos \frac{m\pi x_1}{L} && \text{(case A)} \\ &= \sum_{m=1}^{\infty} W_m \sin \left(m - \frac{1}{2}\right) \frac{\pi x_1}{L} && \text{(case B)} \\ &= \sum_{m=1}^{\infty} W_m \sin \frac{m\pi x_1}{L}. && \text{(case C)} \end{aligned}\right\} \qquad (19)$$

The coefficients W_0, W_m here are functions of x_2, x_3.

* Department of Mathematics, Imperial College, London.

The first derivative of $\omega(x_1)$ is expanded similarly:

$$\left.\begin{aligned} \omega' &= \Omega_0 + \sum_{m=1}^{\infty} \Omega_m \cos\frac{m\pi x_1}{L} && \text{(A)} \\ &= \sum_{m=1}^{\infty} \Omega_m \sin\left(m - \frac{1}{2}\right)\frac{\pi x_1}{L} && \text{(B)} \\ &= \sum_{m=1}^{\infty} \Omega_m \sin\frac{m\pi x_1}{L}. && \text{(C)} \end{aligned}\right\} \quad (20)$$

The coefficients Ω_0, Ω_m are of course constants.

When these expansions are inserted into Eqns (7), (8) and (9) the outcome, in view of the orthogonality of the trigonometrical functions, is:

$$\left.\begin{aligned} \nabla^2 W_0 &= 0, && \text{(A)} \\ \nabla^2 W_m - \frac{m^2\pi^2 E}{L^2 G} W_m &= 0, && \text{(A or C)} \\ \nabla^2 W_m - \left(m - \frac{1}{2}\right)^2 \frac{\pi^2 E}{L^2 G} W_m &= 0, && \text{(B)} \end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned} \partial W_0/\partial n &= -t\Omega_0 \text{ on the boundary} && \text{(A)} \\ \partial W_m/\partial n &= -t\Omega_m \text{ on the boundary} && \text{(A, B or C),} \end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned} &\int_A W_0 \,\mathrm{d}A = 0, && \text{(A)} \\ &\int_A W_m \,\mathrm{d}A = 0 \quad (m = 1, 2, \ldots). && \text{(A, B or C)} \end{aligned}\right\} \quad (23)$$

We notice that if Ω_0 or any Ω_m should be zero then W_0 or the corresponding W_m is also zero for all x_2, x_3. For nonzero Ω_0 or Ω_m we can introduce functions $f_0(x_2, x_3)$, $f_m(x_2, x_3)$ defined by $f_0 = W_0/\Omega_0$, $f_m = W_m/\Omega_m$, whose value will be *independent of the torque* and given by:

$$\left.\begin{aligned} &\nabla^2 f_m - \frac{m^2\pi^2 E}{L^2 G} f_m = 0, && \text{(A or C)} \\ &\nabla^2 f_m - \frac{(m-\frac{1}{2})^2\pi^2 E}{L^2 G} f_m = 0, && \text{(B)} \\ &\frac{\partial f_m}{\partial n} = -t \quad \text{on the boundary,} && \text{(A, B or C)} \\ &\int_A f_m \,\mathrm{d}A = 0, && \text{(A, B or C)} \end{aligned}\right\} \quad (24)$$

where $m = 1, 2, \ldots$ and, in Case A, $m = 0$ also. The solution of Eqns (24) will usually require recourse to numerical methods.

The first torsion problem

With $\omega(x_1)$ given, the constants Ω_0, Ω_m can be found from Eqns (20) by standard orthogonality methods, and the nonzero W_0, W_m functions are given by $W_0 = \Omega_0 f_0$, $W_m = \Omega_m f_m$. The torque involved then follows from Eqn (10).

Let

$$H_m = \oint f_m t \,\mathrm{d}s, \quad (25)$$

$$\left.\begin{aligned} \tau_m(x_1) &= \cos\frac{m\pi x_1}{L}, && m = 0, 1, 2, \ldots, && \text{(A)} \\ &= \sin\left(m - \frac{1}{2}\right)\frac{\pi x_1}{L}, && m = 1, 2, \ldots, && \text{(B)} \\ &= \sin\frac{m\pi x_1}{L}, && m = 1, 2, \ldots. && \text{(C)} \end{aligned}\right\} \quad (26)$$

Equation (10) becomes:

$$T(x_1) = G\sum_m (J + H_m)\Omega_m \tau_m(x_1). \qquad \text{(A, B or C)} \tag{27}$$

We notice that if $H_m \ll J$ for all m, then Eqn (27) approaches $T(x_1) = GJ\omega'$; and that for Saint-Venant torsion (Case A with $\omega' = \text{constant} = \Omega_0$), $T = G(J + H_0)\omega'$, since $\tau_0 = 1$, so that $J + H_0$ corresponds to the Saint-Venant torsion constant.

The second torsion problem

With $T(x_1)$ given, the constants Ω_0, Ω_m are found from Eqn (27):

$$\left.\begin{aligned} \Omega_m &= \frac{2\int_0^L T\tau_m \, dx_1}{GL(J + H_m)}, \qquad \text{(A, B or C)} \\ \Omega_0 &= \frac{\int_0^L T \, dx_1}{GL(J + H_0)}. \qquad \text{(A)} \end{aligned}\right\} \tag{28}$$

The rotation $\omega(x_1)$ is obtained by integrating Eqn (20), and the solution is complete, since the warping functions are again given by $W_m = \Omega_m f_m$.

A common case arises when the torque results from a distributed load $q(x_1)$ at an eccentricity $e(x_1)$. For equilibrium we have $T' = qe$, but the torque itself may be statically indeterminate:

$$T(x_1) = T_0 + \int_0^{x_1} qe \, dx_1,$$

where T_0 is initially unknown. In this event we invoke the compatibility condition (11) together with the expansions (20):

$$\omega(L) - \omega(0) = \int_0^L \omega' \, dx_1 = \sum_m \Omega_m \int_0^L \tau_m \, dx_1 = 0.$$

From the definitions (26) of τ_m we thus find:

$$\left.\begin{aligned} \Omega_0 &= 0, \qquad \text{(A)} \\ \sum_{m=1}^{\infty} \frac{\Omega_m}{m - \frac{1}{2}} &= 0, \qquad \text{(B)} \\ \sum_{m=1,3,\ldots}^{\infty} \frac{\Omega_m}{m} &= 0, \qquad \text{(C)} \end{aligned}\right\} \tag{29}$$

from which to derive the statically indeterminate end-torque.

Example

Consider the case where qe is constant, giving a linearly varying torque $T = T_0 + qe\,x_1$. The constants Ω_0, Ω_m are found from Eqns (28).

Case A:

$$\Omega_0 = \frac{T_0 + \frac{1}{2}qeL}{G(J + H_0)},$$

$$\Omega_m = \frac{-4\,qeL}{m^2\pi^2 G(J + H_m)} \qquad (m \text{ odd}),$$

$$= 0 \qquad (m \text{ even}).$$

Equation (29) shows that $T_0 = -\frac{1}{2}qeL$.

Case B:

$$\Omega_m = \frac{2}{(m-\frac{1}{2})\pi G(J+H_m)}\left\{T_0 - \frac{(-1)^m qeL}{(m-\frac{1}{2})\pi}\right\}.$$

The compatibility condition (29) now gives:

$$T_0 \sum_{n=1}^{\infty} \frac{1}{(n-\frac{1}{2})^2 (J+H_n)} = \frac{qeL}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-\frac{1}{2})^3 (J+H_n)},$$

from which T_0 is to be calculated and substituted in the formula just obtained for Ω_m.
Case C:

$$\Omega_m = \frac{2(2T_0 + qeL)}{m\pi G(J+H_m)} \quad (m \text{ odd}),$$

$$= -\frac{2qeL}{m\pi G(J+H_m)} \quad (m \text{ even}).$$

Again, Eqn (29) shows that $T_0 = -\frac{1}{2}qeL$ (as would be expected from the symmetry of the problem), and it follows that:

$$\Omega_m = 0 \quad (m \text{ odd}).$$

NUMERICAL CALCULATIONS

It is seen from Eqns (17) and (24) that, whichever method of solution is adopted, the two-dimensional problem to be solved can be expressed in the form:

$$\nabla^2 f - Cf = X, \tag{30}$$

where $\partial f/\partial n$ has given values on the boundary, C is a known constant, $X(x_2, x_3)$ takes known values throughout the domain, and the condition $\int_A f \, dA = 0$ is satisfied by the required anti-symmetry of f.

A computer program written to solve this problem, by Successive Over-Relaxation of finite difference approximations using a variable rectangular mesh, had as its primary output the integrals K_m or $J + H_m$; it was applied to members having the cross-sections of Fig. 4, with $E/G = 2.6$ and $L/d = 20$. (Note that the values of H_m, but not of K_m, depend upon the ratio L/d.) The cross-sections were designed so that the first would fall indisputably within the thin-wall domain for which previous theories were intended, the second would have proportions typical of many pre-stressed concrete members, for which the thin-wall approximation might be thought questionable, and the third would have dimensions for which the earlier theories were clearly not appropriate.

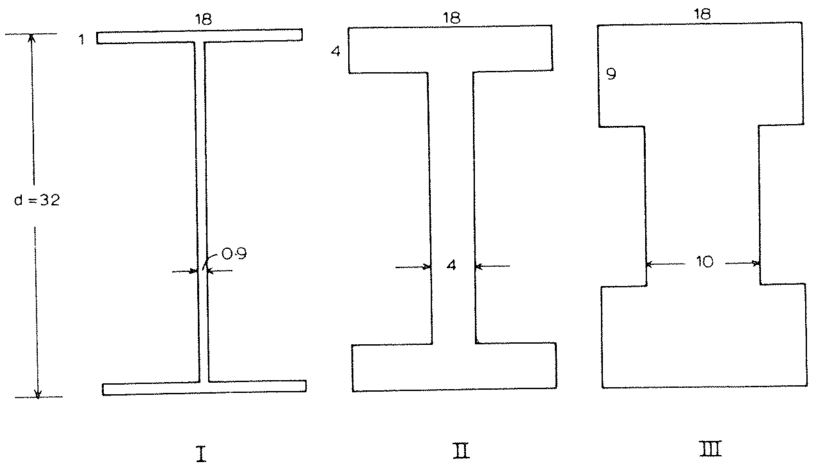

FIG. 4. Cross-sections used for numerical examples.

TABLE 1. NUMERICAL RESULTS FOR THE CROSS-SECTIONS OF FIG. 4

m	K_m	$J + H_m$ Case A or C	$J + H_m$ Case B
Thin section			
0		*$18.34\ d^4 \times 10^{-6}$	
1	$18.340\ d^4 \times 10^{-6}$	32.26	$21.82\ d^4 \times 10^{-6}$
2		73.67	49.57
3	$-5.651\ d^6 \times 10^{-4}$ $-5.665\ d^6 \times 10^{-4}$ (Bleich's formula)	141.60	104.40
4		234.50	185.05
5	$-4.584\ d^8 \times 10^{-5}$	350.29	289.68
6		486.55	416.03
7	$-3.764\ d^{10} \times 10^{-6}$	640.47	561.44
8		809.34	723.23
9	$-3.094\ d^{12} \times 10^{-7}$		
11	$-2.544\ d^{14} \times 10^{-8}$		
Medium section			
0		*$1.2475\ d^4 \times 10^{-3}$	
1	$12.475\ d^4 \times 10^{-4}$	1.2917	$1.2586\ d^4 \times 10^{-3}$
2		1.4231	1.3466
3	$-1.792\ d^6 \times 10^{-3}$ $-1.906\ d^6 \times 10^{-3}$ (Bleich's formula)	1.6387	1.5206
4		1.9340	1.7768
5	$-1.370\ d^8 \times 10^{-4}$	2.3028	2.1096
6		2.7377	2.5125
7	$-1.069\ d^{10} \times 10^{-5}$	3.2304	2.9773
8		3.7723	3.4958
9	$-8.347\ d^{12} \times 10^{-7}$		
11	$-6.522\ d^{14} \times 10^{-8}$		
Thick section			
0		*$1.3827\ d^4 \times 10^{-2}$	
1	$13.827\ d^4 \times 10^{-3}$	1.3878	$1.3840\ d^4 \times 10^{-2}$
2		1.4032	1.3943
3	$-2.090\ d^6 \times 10^{-3}$ $-3.175\ d^6 \times 10^{-3}$ (Bleich's formula)	1.4284	1.4146
4		1.4630	1.4446
5	$-1.450\ d^8 \times 10^{-4}$	1.5064	1.4837
6		1.5576	1.5310
7	$-1.039\ d^{10} \times 10^{-5}$	1.6160	1.5860
8		1.6804	1.6475
9	$-7.489\ d^{12} \times 10^{-7}$		
11	$-5.404\ d^{14} \times 10^{-8}$		

*Case A only.

The results, together with the value of the restrained-warping constant $E\Gamma/G$ as given by Bleich's formula [12], are given in Table 1. The agreement between the value K_3 and Bleich's constant is very close for the thin-walled section; it is within 6.5% for the second section; and, as was to be expected, there is no agreement in the third case.

AN ECCENTRICALLY LOADED BEAM

As an example of the importance of torsion in beams, and of the application of the present results, we consider the thin-walled member treated above to be mounted over two equal spans and to carry a uniformly distributed load q as shown in Fig. 5. The problem is two degrees statically indeterminate, and we shall make our goal the values of the vertical

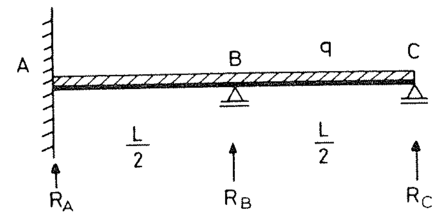

FIG. 5. A two-span beam example.

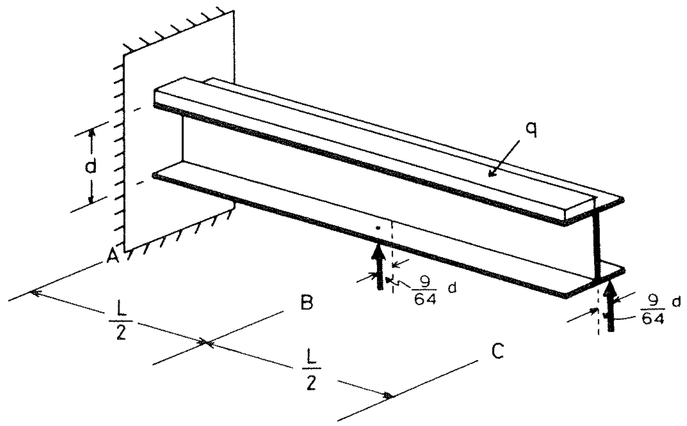

FIG. 6. The two-span beam eccentrically loaded and supported.

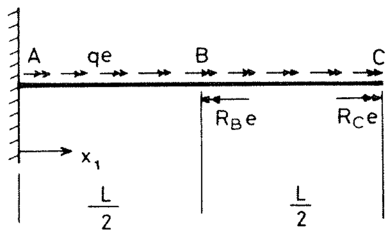

FIG. 7. The variable torsion loading.

support reactions. If the member is both loaded and supported concentrically, we have a purely flexural problem, and elementary methods show that the support forces are:

$$R_A = 0.2321\,qL, \quad R_B = 0.5715\,qL, \quad R_C = 0.1964\,qL. \tag{31}$$

If, however, the load and support forces at B and C are all applied with an eccentricity e of one quarter of the flange width (so that $e = 9/64\,d$), as shown in Fig. 6, then we have a problem of combined flexure and nonuniform torsion, in which the vertical displacements of the centroids will be given by:

$$v_B = -e\omega_B, \quad v_C = e\omega_C \quad (v = -u_3).$$

The problem is best treated by the superposition of the torsion problem of Fig. 7 and the flexure problem of Fig. 5, whose elementary solution is now:

$$\begin{bmatrix} R_B \\ R_C \end{bmatrix} = \begin{bmatrix} 0.5715 \\ 0.1964 \end{bmatrix} qL - \begin{bmatrix} 109.71 & -34.29 \\ -34.29 & 13.71 \end{bmatrix} \begin{bmatrix} -e\omega_B \\ e\omega_C \end{bmatrix} \frac{EI}{L^3}, \tag{32}$$

where I is the relevant second moment of area of the cross-section ($= 0.01018d^4$). The torsion problem will be solved by both methods.

Trigonometric series solution

The torque can be found from equilibrium as:

$$\left.\begin{aligned} T &= qe(L - x_1) - R_B e + R_C e \quad (x_1 < L/2) \\ &= qe(L - x_1) + R_C e \qquad\qquad (x_1 > L/2) \end{aligned}\right\} \tag{33}$$

and when these expressions, together with the values of $(J + H_m)$ from Table 1, are inserted in Eqn (28) for a Case B problem we find:

$$\Omega_m = \sum_{j=1}^{3} \bar{\Omega}_{mj} X_j, \tag{34}$$

where:

$$X_j = \left(\frac{qeL}{Gd^4}, \frac{R_B e}{Gd^4}, \frac{R_C e}{Gd^4}\right)$$

and $\bar{\Omega}_{mj}$ is listed in Table 2.

The rotation ω, which is zero at $x_1 = 0$, can be expressed as $\int_0^{x_1} \omega' \, dx_1$ and it follows from Eqn (20) that:

$$\omega_B = \frac{L}{\pi} \sum_{m=1}^{\infty} \frac{\Omega_m}{m - \frac{1}{2}} \left[1 - \cos\left(m - \frac{1}{2}\right)\frac{\pi}{2}\right],$$

$$\omega_C = \frac{L}{\pi} \sum_{m=1}^{\infty} \frac{\Omega_m}{m - \frac{1}{2}}.$$

If we denote the sum of the first M terms of these series by $\omega^{(M)}$ and use the values found in Eqn (34) to give:

$$\omega_B^{(M)} = \sum_{j=1}^{3} \bar{\omega}_{Bj}^{(M)} X_j L, \quad \omega_C^{(M)} = \sum_{j=1}^{3} \bar{\omega}_{Cj}^{(M)} X_j L, \tag{35}$$

then we can inspect the convergence in Table 3. Three-figure accuracy—enough for most engineering purposes—is achieved in about six terms, 1% accuracy in three.

TABLE 2. NUMERICAL VALUES FOR EQN (34)

m	$\bar{\Omega}_{m1}$	$\bar{\Omega}_{m2}$	$\bar{\Omega}_{m3}$
1	2.1200×10^4	-1.7088×10^4	5.8341×10^4
2	1.0379×10^4	-1.4616×10^4	8.5619×10^3
3	2.1285×10^3	-4.1638×10^3	2.4391×10^3
4	1.0723×10^3	-2.8790×10^2	9.8295×10^2
5	4.5383×10^2	-1.4304×10^2	4.8837×10^2
6	2.9432×10^2	-4.7496×10^2	2.7822×10^2
7	1.6590×10^2	-2.9780×10^2	1.7445×10^2
8	1.2235×10^2	-3.4376×10	1.1737×10^2

TABLE 3. CONVERGENCE OF THE COMPONENTS OF EQN (35)

M	$\bar{\omega}_{B1}^{(M)}$	$\bar{\omega}_{B2}^{(M)}$	$\bar{\omega}_{B3}^{(M)}$	$\bar{\omega}_{C1}^{(M)}$	$\bar{\omega}_{C2}^{(M)}$	$\bar{\omega}_{C3}^{(M)}$
1	3.953	− 3.186	10.878	13.496	− 10.878	37.141
2	7.713	− 8.481	13.980	15.699	− 13.980	38.958
3	8.176	− 9.386	14.510	15.970	− 14.510	39.269
4	8.204	− 9.394	14.536	16.067	− 14.536	39.358
5	8.213	− 9.397	14.546	16.099	− 14.546	39.393
6	8.243	− 9.444	14.574	16.116	− 14.574	39.409
7	8.256	− 9.469	14.589	16.125	− 14.589	39.417
8	8.258	− 9.469	14.590	16.130	− 14.590	39.422

All these values must be multiplied by 1000.

The ω_B, ω_C values from Eqn (35) transform Eqn (32) into a pair of equations for R_B and R_C which, together with overall vertical equilibrium, give:

$$R_A = 0.0203\,qL, \quad R_B = 0.9112\,qL, \quad R_C = 0.0685\,qL. \tag{36}$$

Comparison with the figures of Eqn (31) demonstrates the sensitivity of beams of low torsional rigidity to eccentricities in the loading.

Solution by differential equation

If only two terms are retained on its right-hand side, Eqn (18) can readily be solved analytically. With $K_1/|K_3| = \mu^2$, the equation becomes:

$$\omega''' - \mu^2\omega' = -\,T(x_1)/G|K_3|, \tag{37}$$

where $T(x_1)$ takes the values of Eqn (33), discontinuous at $x_1 = L/2$. It can be shown that the solution of Eqn (37) in which ω, ω' and ω'' are all continuous at $x_1 = L/2$ is:

$$\omega = \frac{eL}{GK_1}\left\{(qL - R_B + R_C)\frac{x_1}{L} - qL\left(\frac{x_1^2}{2L^2} + \frac{1}{\mu^2 L^2}\right) + \frac{R_B}{\mu L}\left[\mu\left\{x_1 - \frac{L}{2}\right\} - \sinh\mu\left\{x_1 - \frac{L}{2}\right\}\right]\right\} + A\cosh\mu x_1 + B\sinh\mu x_1 + C, \tag{38}$$

where A, B and C are single-valued constants and the quantity $\{x_1 - L/2\}$ is to be treated as zero if $x_1 < L/2$.

The three constants are used to satisfy the Case B end conditions:

$$\omega = 0 \quad \text{and} \quad \omega' = 0 \quad \text{at} \quad x_1 = 0; \quad \omega'' = 0 \quad \text{at} \quad x_1 = L$$

[Eqns (13) and (14)], and when the values of K_1 and K_3 are inserted from Table 1 we find:

$$\begin{bmatrix}\omega_B \\ \omega_C\end{bmatrix} = \begin{bmatrix}0.8247 & -0.9453 & 1.4580 \\ 1.6124 & -1.4580 & 3.9415\end{bmatrix}\begin{bmatrix}qeL^2/Gd^4 \\ R_B eL/Gd^4 \\ R_C eL/Gd^4\end{bmatrix} \times 10^4. \tag{39}$$

These figures compare directly with those of Eqn (35) and Table 3.

With further terms on the right-hand side of Eqn (18) an analytical solution is cumbersome, and Fig. 8 results from a finite difference numerical integration of the equation, using 96 intervals. Results derived from an odd number of terms show a spurious oscillation

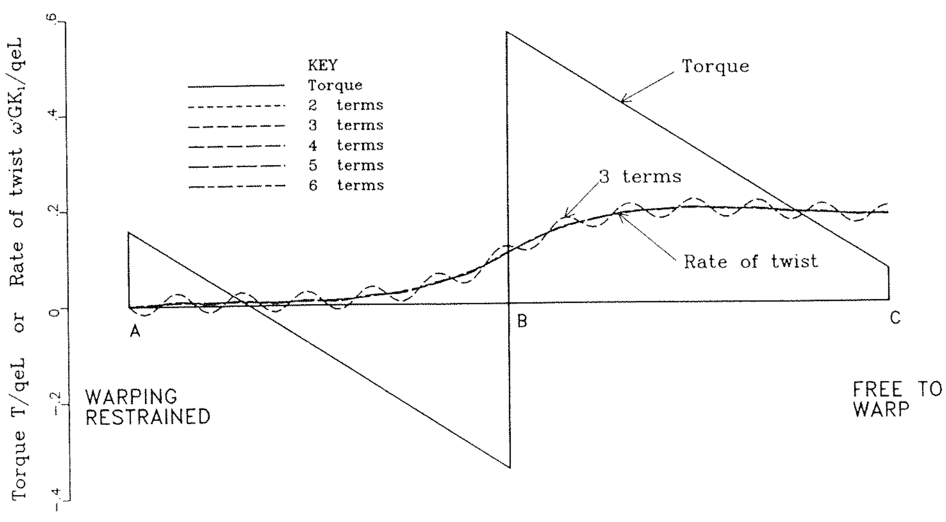

FIG. 8. Results for the problem of Fig. 6.

which will not occur from an even number. In the present case the five-term oscillation is very small, so the two-, four-, five- and six-term results are almost indistinguishable.

Clearly for this thin-walled cross-section two terms are enough; we are unable to assert that this will generally be the case.

CONCLUSIONS

The paper has addressed the following classical problem for straight elastic members with constant doubly symmetric but otherwise arbitrary cross-section:

> *What are the displacements and stresses provoked by any loading which involves a torque, where the torque may vary (continuously or discontinuously) along the member?*

The purely flexural part of the response may be separated and incorporated by superposition, so only the torsional response requires study.

Sokolnikoff [15] discussed the problem, but only for the particular case of a linearly varying torque on a cantilever. The other successful analyses have all been approximations limited to thin-walled members.

The governing equations have been established here [Eqns (7), (8), (10), (13) and (14)] and two quite different methods of solving them have been demonstrated. In their convenience they have different virtues, but they give the same answer to the statically indeterminate example of Fig. 6, and should do so in all cases. This example is a practical engineering one, and with its discontinuous torque it cannot be solved by Sokolnikoff's method. The solution is illustrated in Fig. 8, and a comparison of the support reactions of Eqns (31) and (36) reveals the little-appreciated sensitivity of torsionally weak beams to eccentricity of loading and/or support—an important immediate result from the new analysis.

The differential equation method of solution involves calculating a constant K_3. Existing thin-wall approximations depend essentially on a constant Γ, for which Bleich [12] gives a formula. For very thin walls, $\Gamma = -(G/E)K_3$, so a direct comparison can be made with the present theory. It transpires, as expected, that for very thin walls (height/thickness = 32) agreement is close, but at lower values (e.g. height/thickness = 8) Bleich's values lose their accuracy (see Table 1). In the absence of a complete analysis it has been impossible until now to assess the accuracy of the widely used approximate theory.

REFERENCES

1. C. WEBER, Die Lehrer der Drehungsfestigheit. *Forscharb. Ing. Wes.* **249**, 60–62 (1921).
2. M. S. G. CULLIMORE, The shortening effect. *Engineering Structures*, pp. 153–163. Butterworths, London (1949).
3. D. G. ASHWELL, The axis of distortion of a twisted prism. *Phil. Mag.* **42**, 820–832 (1951).
4. M. GREGORY, The bending and shortening effect of pure torque. *Austral. J. Appl. Sci.* **11**, 209–216 (1960).
5. J. C. BUCKLEY, The bifilar property of twisted strips. *Phil. Mag.* **28**, 778–787 (1914).
6. H. WAGNER, Torsion and buckling of open sections. NACA Technical Memorandum, No. 807 (1936).
7. M. A. BIOT, Increase of torsional stiffness of a prismatic bar due to axial tension. *J. Appl. Phys.* **10**, 860–864 (1939).
8. J. N. GOODIER, Elastic torsion in the presence of axial stress. *J. Appl. Mech.* **17**, 383–387 (1950).
9. M. GREGORY, Elastic torsion of members of thin open cross-section. *Austral. J. Appl. Sci.* **12**, 174–193 (1961).
10. S. TIMOSHENKO, On the torsion of a prism. *Proc. Lond. Math. Soc.* (*Ser.* 2) **20**, 389–397 (1922).
11. J. N. GOODIER, The buckling of compressed bars by torsion and flexure. *Cornell Engng Exp. Stu. Bull.*, No. 27 (1941).
12. F. BLEICH, *Buckling Strength of Metal Structures*. McGraw-Hill, New York and London (1952).
13. V. Z. VLASOV, *Thin-walled Elastic Beams* (Transl. from Russian). Israel Program for Scientific Translations, Jerusalem (1961).
14. E. REISSNER, On non-uniform torsion of cylindrical rods. *J. Math. Phys.* **31**, 214–221 (1952).
15. I. S. SOKOLNIKOFF, *Mathematical Theory of Elasticity*, 2nd Edn. McGraw-Hill, New York and London (1956).
16. S. J. MCMINN, Linear theory of non-uniform torsion. *J. Mech. Engng Sci.* **3**, 80–88 (1961).
17. C. J. BURGOYNE, The non-linear behaviour of elastic beams and beam-columns. Ph.D. Thesis, London University, London, U.K. (1981).