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CALCULATION OF ELASTO-PLASTIC RIGIDITIES USING THE EXACT ILYUSHIN YIELD SURFACE

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Abstract—The analysis of failure loads and mechanisms of plated structures requires knowledge of the elasto-plastic rigidities when elements of the structure are fully plastic. Previous attempts to calculate these rigidities from a single-layer analysis have given results which differ from those obtained by multi-layer analysis. It is shown that by using an exact representation of the Ilyushin yield surface, and by considering the normal direction to that surface, it is possible to obtain a unique stress distribution through the thickness of the plate, which allows the correct value of the rigidities to be found. The reasons for the discrepancy are discussed, and it is concluded that it arises from incompatible assumptions regarding the relative magnitudes of the elastic and plastic strain components. A numerical example is given to illustrate the phenomenon.

NOMENCLATURE

The principal notation used in the paper is described below. Some other notation that is used only once is described where it is used.

$\mathbf{e}$	strain vector at any position
f	yield function in stress space (von Mises)
$\mathbf{m}$	non-dimensional flexural stress resultant vector
m_x, m_y, m_{xy}	non-dimensional flexural stress resultant components
$\mathbf{n}$	non-dimensional in-plane stress resultant vector
n_x, n_y, n_{xy}	non-dimensional in-plane stress resultant components
z	non-dimensional position through thickness of plate
$\mathbf{A}$	matrix relating stress vector to plastic strain increment vector [eqn (27)]
C	constant defining magnitude of normal vector
$\mathbf{B}^*, \mathbf{C}^*, \mathbf{D}^*$	elasto-plastic rigidity sub-matrices [eqn (7)]
$\mathbf{E}$	elastic stiffness matrix
$\mathbf{E}^*$	elasto-plastic rigidity matrix [eqn (21)]
$\mathbf{E}_m, \mathbf{E}_{mn}, \mathbf{E}_n$	see eqn (41)
F	yield function in stress resultant space
F_1, F_2, F_3	see eqns (35) (36)
F_t, F_{tm}, F_m	components of normal to yield surface
$L_0 \Rightarrow L_4$	through-thickness integrals [eqns (44), (45)]
$P_\varepsilon, P_{\varepsilon\kappa}, P_\kappa$	quadratic strain intensity
Q_1, Q_{12}, Q_2	quadratic stress resultant (not on yield surface)
Q_t, Q_{tm}, Q_m	quadratic stress resultant on the yield surface
R	defined in eqn (42)
S	Young's modulus
α, β, γ	parameter of the present formulation
$\alpha_0, \beta_0, \gamma_0$	initial estimates of α, β and γ
$\boldsymbol{\varepsilon}$	mid-plane strain as vector
$\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$	mid-plane strain components
$\boldsymbol{\kappa}$	non-dimensional curvature vector
$\kappa_x, \kappa_y, \kappa_{xy}$	non-dimensional curvature components
λ	plastic strain rate multiplier
η	multiplier relating given set of Q_t to the yield surface
η_0	initial estimate of η
ψ, ω	defined in eqn (42)
$\boldsymbol{\sigma}$	non-dimensional stress vector.

INTRODUCTION

The analysis of failure loads and failure mechanisms for plated structures requires knowledge of the elasto-plastic rigidities when elements of the structure are fully plastic. These rigidities can be calculated accurately by performing a multi-layer analysis, which requires knowledge of the stress distribution through the thickness of the plate, but this requires extensive computing resources, both of time and storage. It is more efficient to calculate the rigidities from a single layer analysis, which relates to stress resultants. Rigidities calculated by these two methods in the past did not agree, but the reasons for this were not fully explored.

In a companion paper (Burgoyne and Brennan, 1993), the present authors have presented a procedure which reanalyses Ilyushin's exact yield surface defined in terms of stress resultants, using new parameters. Unlike those used in Ilyushin's original derivation, these parameters allow the exact surface [as opposed to an approximation to it, such as those described by Robinson (1971)] to be used in structural calculations. This allows the direction of the normal to the stress resultant yield surface to be calculated accurately.

The normal direction in stress resultant space can be used as the basis of a calculation of the elasto-plastic rigidities. Traditionally, these rigidities have been determined by assuming that any deformation applied to a plate which is at a point on the full plasticity surface consists of an elastic component, which corresponds to a movement in stress resultant space around the yield surface, and a plastic strain component normal to the yield surface. In this paper, it will be demonstrated that this analysis does not give the correct rigidities. Instead, it will be shown that the normal direction to the surface defines the direction of the plastic strain at each layer in the plate. This direction must be normal to the *stress* yield surface at that layer, thus defining the stress throughout the plate, and allowing the calculation of the elasto-plastic rigidities as though a multi-layer analysis had been performed.

Reasons for the differences between the two methods will be discussed, and a numerical example will be given which demonstrates the differences, and also shows how the new parametric description of the surface allows efficient calculation of both the position and the normal direction to the yield surface.

NOTATION

Burgoyne and Brennan (1993) showed how three quadratic stress intensities Q_t, Q_m and Q_{tm} are defined in terms of six non-dimensional stress resultants n_x, n_y, n_{xy}, m_x, m_y and m_{xy} by:

$$\begin{aligned} Q_t &= n_x^2+n_y^2-n_xn_y+3n_{xy}^2, \\ Q_{tm} &= m_xn_x+m_yn_y-\tfrac{1}{2}(m_xn_y+m_yn_x)+3m_{xy}n_{xy}, \\ Q_m &= m_x^2+m_y^2-m_xm_y+3m_{xy}^2. \end{aligned} \tag{1}$$

Q_t, Q_{tm}, Q_m will be taken to refer to points on the yield surface. The corresponding variables Q_1, Q_{12}, Q_2 will relate to a general point in Q-space not on the yield surface.

The corresponding non-dimensionalized mid-plane strains (ε_x, ε_y, ε_{xy}) and curvatures (κ_x, κ_y, κ_{xy}) were defined, from which quadratic strain intensities were obtained:

$$\begin{aligned} P_\varepsilon &= \mathrm{d}\varepsilon_x^2+\mathrm{d}\varepsilon_x\,\mathrm{d}\varepsilon_y+\mathrm{d}\varepsilon_y^2+0.25\,\mathrm{d}\varepsilon_{xy}^2, \\ P_{\varepsilon\kappa} &= 4(\mathrm{d}\varepsilon_x\,\mathrm{d}\kappa_x+0.5(\mathrm{d}\varepsilon_x\,\mathrm{d}\kappa_y+\mathrm{d}\varepsilon_y\,\mathrm{d}\kappa_x)+\mathrm{d}\varepsilon_y\,\mathrm{d}\kappa_y+0.25\,\mathrm{d}\varepsilon_{xy}\,\mathrm{d}\kappa_{xy}), \\ P_\kappa &= 16(\mathrm{d}\kappa_x^2+\mathrm{d}\kappa_x\,\mathrm{d}\kappa_y+\mathrm{d}\kappa_y^2+0.25\,\mathrm{d}\kappa_{xy}^2). \end{aligned} \tag{2}$$

In this paper, it will be possible to work in terms of three component vectors. Thus, in stress resultant space:

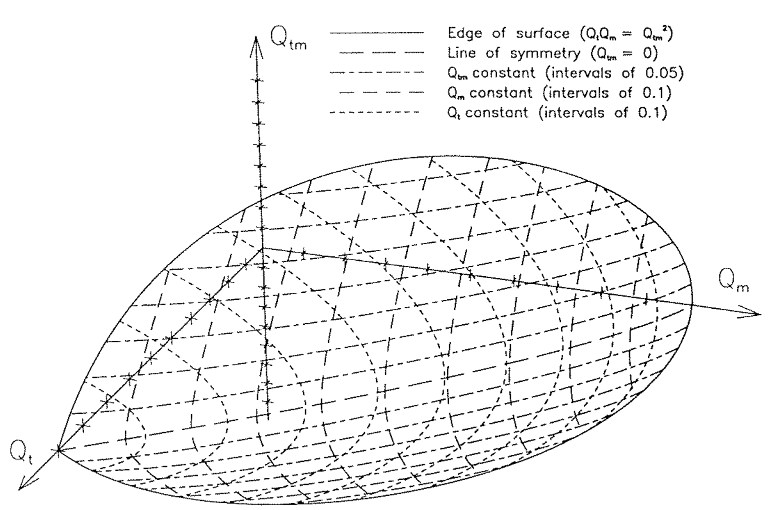

Fig. 1. Three-dimensional view of exact Ilyushin yield surface.

$$\mathbf{n} = (n_x, n_y, n_{xy}) \quad \text{and} \quad \mathbf{m} = (m_x, m_y, m_{xy}),$$
$$\boldsymbol{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_{xy}) \quad \text{and} \quad \boldsymbol{\kappa} = (\kappa_x, \kappa_y, \kappa_{xy}). \tag{3}$$

[The true "engineering" strains and curvatures are given by $\varepsilon_x\sigma_0(1-\nu^2)/S$ and $\kappa_x 4\sigma_0(1-\nu^2)/(St)$ respectively, where S is Young's Modulus.]

The exact Ilyushin full plasticity yield surface will be denoted by F. Position through the thickness of the plate will be defined by z, with the surfaces of the plate defined by $z = \pm\frac{1}{2}$.

In stress space, the vectors defining stress and strain will be given by

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_{xy}) \quad \text{and} \quad \mathbf{e} = (e_x, e_y, e_{xy}) \tag{4}$$

and the corresponding von Mises' yield surface will be denoted by f.

The Ilyushin yield surface (Fig. 1) can be represented in (Q_t, Q_{tm}, Q_m) space as a function of two independent parameters α and β, and a dependent parameter γ (Burgoyne and Brennan, 1993). The parameters are defined as

$$\alpha = \frac{P_\varepsilon}{P_\kappa}, \quad \beta = -\frac{P_{\varepsilon\kappa}}{P_\kappa}, \quad \gamma = \alpha - \beta^2. \tag{5}$$

The surface is symmetrical about the line $Q_{tm} = 0$, and is bounded at the edges by the Schwarz inequality:

$$Q_t Q_m \geqslant Q_{tm}^2. \tag{6}$$

The surface is everywhere smooth, except at one corner where $Q_t = 1$, where there is a discontinuity of slope. The surface has been described in detail in Burgoyne and Brennan (1993), where methods have been described by which points on the surface can be derived as a linear multiple of a general point in Q-space, and the corresponding normal direction to the surface calculated. These methods are used extensively in this paper.

ELASTO-PLASTIC RIGIDITIES

For the analysis of plates and shells, it is necessary to determine the elasto-plastic rigidities, which relate the changes in the mid-plane strains and curvatures to changes in the stress resultants. The relationship that is sought is of the form

$$\begin{bmatrix} \mathbf{dn} \\ \mathbf{dm} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^* & \mathbf{B}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{bmatrix} \begin{bmatrix} \mathrm{d}\boldsymbol{\varepsilon} \\ \mathrm{d}\boldsymbol{\kappa} \end{bmatrix}. \tag{7}$$

Two approaches to the calculation of these rigidities will be considered; one, due to Crisfield (1973), determines them by considering the normal direction to the yield surface itself. The other, to be presented here, derives the rigidities from the stresses within the plate, but eventually is related to the properties of the yield surface. The three sub-matrices **B***, **C*** and **D*** should all be symmetrical and indefinite, but in the first method the **B*** matrix is not. The new method leads to an improved formulation which satisfies the symmetry conditions.

(i) *Traditional derivation*

This approach to the derivation of the elasto-plastic rigidities, used by a number of researchers (Crisfield, 1973; Frieze, 1975; Bieniek and Funaro, 1976; Eggers and Kroplin, 1978; Dinis and Owen, 1982), depends only on the stress resultants **n** and **m**.

For plastic flow to take place, the stress resultants must remain on the yield surface F:

$$\mathrm{d}F = \frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}} \mathrm{d}\mathbf{n} + \frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}} \mathrm{d}\mathbf{m} = 0. \tag{8}$$

It is then assumed that the general normality law holds in stress resultant space, so that

$$\mathrm{d}\boldsymbol{\varepsilon}_{\mathrm{p}} = \lambda \frac{\partial F}{\partial \mathbf{n}} \quad \text{and} \quad \mathrm{d}\boldsymbol{\kappa}_{\mathrm{p}} = \lambda \frac{\partial F}{\partial \mathbf{m}}. \tag{9}$$

The incremental form of Hooke's Law then gives

$$\mathrm{d}\mathbf{n} = \mathbf{E}\left(\mathrm{d}\boldsymbol{\varepsilon} - \lambda \frac{\partial F}{\partial \mathbf{n}}\right), \quad \mathrm{d}\mathbf{m} = \frac{4}{3}\mathbf{E}\left(\mathrm{d}\boldsymbol{\kappa} - \lambda \frac{\partial F}{\partial \mathbf{m}}\right), \tag{10}$$

where **E** is the non-dimensional elastic stiffness matrix

$$\mathbf{E} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}. \tag{11}$$

The multiplier λ may be found by inserting eqn 10 into the tangency condition (8), giving

$$\lambda = \frac{1}{R}\left(\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}} \mathbf{E}\,\mathrm{d}\boldsymbol{\varepsilon} + \frac{4}{3}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}} \mathbf{E}\,\mathrm{d}\boldsymbol{\kappa}\right), \tag{12}$$

where

$$R = \frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}} \mathbf{E} \frac{\partial F}{\partial \mathbf{n}} + \frac{4}{3}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}} \mathbf{E} \frac{\partial F}{\partial \mathbf{m}}. \tag{13}$$

Substitution of these values into eqns (10) leads to expressions for the **B***, **C*** and **D*** matrices:

$$\mathbf{B}^* = -\frac{4}{3R}\mathbf{E}\frac{\partial F}{\partial \mathbf{n}}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}}\mathbf{E},$$

$$\mathbf{C}^* = \mathbf{E} - \frac{1}{R}\mathbf{E}\frac{\partial F}{\partial \mathbf{n}}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}}\mathbf{E},$$

$$\mathbf{D}^* = \frac{4}{3}\mathbf{E} - \frac{16}{9R}\mathbf{E}\frac{\partial F}{\partial \mathbf{m}}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}}\mathbf{E}. \quad (14)$$

The resulting 6×6 matrix is singular, and an applied strain increment in a direction normal to the yield surface causes no change in the stress resultants. However, the $\mathbf{B}^*$ matrix is not itself symmetrical, which it should be since it is a stiffness matrix [see eqn (24) below]. Symmetry is preserved only when

$$\frac{\partial F}{\partial \mathbf{n}} = C\frac{\partial F}{\partial \mathbf{m}}, \quad (15)$$

which occurs along $Q_t Q_m = Q_{tm}^2$. Then,

$$R = (C^2 + 4/3)\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}}\mathbf{E}\frac{\partial F}{\partial \mathbf{m}} \quad (16)$$

and

$$\mathbf{B}^* = -\frac{4}{3}C\mathbf{E}_p,$$

$$\mathbf{C}^* = \mathbf{E} - C^2\mathbf{E}_p,$$

$$\mathbf{D}^* = \frac{4}{3}\mathbf{E} - \frac{16}{9}\mathbf{E}_p, \quad (17)$$

where

$$\mathbf{E}_p = \frac{1}{R}\mathbf{E}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}}\frac{\partial F}{\partial \mathbf{m}}\mathbf{E}. \quad (18)$$

The condition given by eqn (15) corresponds to a rectangular stress block each side of the depth given by $z = \beta$. For a rectangular stress distribution $\mathbf{B}^*$ must be zero, which only occurs if C is zero. Under these circumstances, $\mathbf{C}^* = \mathbf{E}$, which is clearly an error.

The reason for this discrepancy relates to one of the assumptions made in the original derivation of the yield surface: The strains were assumed to be large, and thus wholly plastic.

Consider a point A on the stress resultant yield surface; there must be a corresponding very large strain resultant distribution normal to the yield surface at that point. At an infinitesimally adjacent point B on the yield surface, there must be a similarly large strain resultant distribution normal to the surface at B. An (infinitesimal) movement in stress resultant space from A to B must be accompanied by a change in the strain resultant equal to the difference between the two strain resultant states. This change will *not* be small.

When considering movement around the yield surface in *stress* space, it is assumed that the change in strain is made up of an elastic component determined from the stress change by Hooke's Law, and a plastic component normal to the yield surface. Both components will be of the same order of magnitude, so stress changes can be associated with strain changes and elasto-plastic stiffnesses derived. As shown in the previous paragraph, when considering *stress resultant* space, the strain resultant components are not of the same order of magnitude, so no such association can be derived and the rigidities cannot be determined by a similar argument.

In effect, this means that the normality rule cannot be applied in stress resultant space when only small movements are being considered, which is the case when deriving local rigidities.

(ii) *Revised formulation*

The new formulation, presented here, does not rely on the assumption that the normality rule applies in stress resultant space, and will be shown to satisfy all requirements of the symmetry conditions. Instead, the normality rule is applied to the stresses at each level within the plate. However, it will be shown that the rigidities can still be related to the properties of the stress resultant yield surface at the point in question, thus fulfilling one of the prime requirements of a single layer analysis, which is that it must be possible to work entirely in terms of stress resultants.

For the plane stress state in which continued plastic deformation is occurring, the modified form of Hooke's Law

$$\mathrm{d}\boldsymbol{\sigma} = \mathbf{E}\left(\mathrm{d}\mathbf{e} - \lambda\frac{\partial f}{\partial \boldsymbol{\sigma}}\right) \tag{19}$$

and the tangency condition

$$\mathrm{d}f = \frac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}}\mathrm{d}\boldsymbol{\sigma} = 0 \tag{20}$$

are sufficient to determine the plastic multiplier λ and to establish a unique relationship between increments of stress $\mathrm{d}\boldsymbol{\sigma}$ and increments of strain $\mathrm{d}\mathbf{e}$ of the form

$$\mathrm{d}\boldsymbol{\sigma} = \mathbf{E}^*\,\mathrm{d}\mathbf{e}, \tag{21}$$

where

$$\mathbf{E}^* = \mathbf{E} - \frac{\mathbf{E}\dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}}\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\mathbf{E}}{\dfrac{\partial f^{\mathrm{T}}}{\partial \boldsymbol{\sigma}}\mathbf{E}\dfrac{\partial f}{\partial \boldsymbol{\sigma}}}. \tag{22}$$

This process was first obtained by Yamada *et al.* (1968). $\mathbf{E}^*$ is termed the elasto-plastic modular matrix and replaces the elastic matrix $\mathbf{E}$. It is a symmetric and singular 3×3 matrix, and for an ideal elastic-plastic material is dependent on the current stress level $\boldsymbol{\sigma}$.

The assumption that the strain varies linearly through the thickness can be expressed in incremental form

$$\mathrm{d}\mathbf{e} = \mathrm{d}\boldsymbol{\varepsilon} + 4z\,\mathrm{d}\boldsymbol{\kappa} \tag{23}$$

and defining the stress resultant increment as

$$\mathrm{d}\mathbf{n} = \int \mathrm{d}\boldsymbol{\sigma}\,\mathrm{d}z, \quad \mathrm{d}\mathbf{m} = 4\int z\,\mathrm{d}\boldsymbol{\sigma}\,\mathrm{d}z \tag{24}$$

allows the rigidities in eqn (7) to be found.

The elasto-plastic tangential rigidities $\mathbf{B}^*$, $\mathbf{C}^*$ and $\mathbf{D}^*$ are defined by

$$\mathbf{C}^* = \int_{-1/2}^{1/2} \mathbf{E}^* \, dz,$$

$$\mathbf{B}^* = 4\int_{-1/2}^{1/2} z\mathbf{E}^* \, dz,$$

$$\mathbf{D}^* = 16\int_{-1/2}^{1/2} z^2\mathbf{E}^* \, dz. \tag{25}$$

In general, these integrals cannot be explicitly determined, but for the case of a stress resultant distribution that satisfies the Ilyushin criterion, an exact solution may be obtained.

Consider a given set of stress resultsnts ($\mathbf{n}$, $\mathbf{m}$). The parameters β and γ may be determined by the methods given in the earlier paper (Burgoyne and Brennan, 1993), and the normals $\partial f/\partial \mathbf{n}$ and $\partial f/\partial \mathbf{m}$ determined.

From the normality law, the unique stress distribution may be determined. This is a critical step, as it allows stress resultant changes to be related to stress changes.

The strict mathematical form of the normality law in stress resultant space is

$$\begin{bmatrix} d\boldsymbol{\varepsilon} \\ d\boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} d\boldsymbol{\varepsilon}_p \\ d\boldsymbol{\kappa}_p \end{bmatrix} = \lambda_1 \begin{bmatrix} \partial F/\partial \mathbf{n} \\ \partial F/\partial \mathbf{m} \end{bmatrix}, \tag{26}$$

since the assumption has already been made that the elastic component is small by comparison with the plastic component.

In stress space, using von Mises' yield criterion, the stress state can be related to the change in the plastic strain component (and hence, in the present case, the change in the total strain component), by

$$\boldsymbol{\sigma} = \frac{1}{3\lambda_2(z)} \mathbf{A} \, d\mathbf{e}_p = \frac{1}{3\lambda_2(z)} \mathbf{A} \, d\mathbf{e}, \tag{27}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \tag{28}$$

and

$$\lambda_2(z) = \frac{1}{\sqrt{3}} \sqrt{de_x^2 + de_x \, de_y + de_y^2 + 0.25 \, de_{xy}^2}. \tag{29}$$

The standard linear assumption about strains is made [eqn (23)] and substitution of eqn (26) allows the strain at any level to be related to the normal direction to the stress resultant yield surface

$$d\mathbf{e} = \lambda_1 \left(\frac{\partial F}{\partial \mathbf{n}} + 4z \frac{\partial F}{\partial \mathbf{m}} \right). \tag{30}$$

Thus,

$$\boldsymbol{\sigma} = \frac{\lambda_1}{3\lambda_2(z)} \mathbf{A} \left(\frac{\partial F}{\partial \mathbf{n}} + 4z \frac{\partial F}{\partial \mathbf{m}} \right). \tag{31}$$

It is now necessary to eliminate, or find expressions for, the two λ terms. If eqn (23) is substituted into eqn (29), and the result expanded, an expression for λ_2 is obtained:

$$\lambda_2(z) = \frac{1}{\sqrt{3}}\sqrt{P_\varepsilon + 2zP_{\varepsilon\kappa} + P_\kappa z^2}. \tag{32}$$

Substitution of eqn (26) gives

$$P_\varepsilon = \lambda_1^2\left(\left(\frac{\partial F}{\partial n_x}\right)^2 + \frac{\partial F}{\partial n_x}\frac{\partial F}{\partial n_y} + \left(\frac{\partial F}{\partial n_y}\right)^2 + \frac{1}{4}\left(\frac{\partial F}{\partial n_{xy}}\right)^2\right),$$

$$P_{\varepsilon\kappa} = 4\lambda_1^2\left(\frac{\partial F}{\partial n_x}\frac{\partial F}{\partial m_x} + \frac{1}{2}\left(\frac{\partial F}{\partial n_x}\frac{\partial F}{\partial m_y} + \frac{\partial F}{\partial n_y}\frac{\partial F}{\partial m_x}\right) + \frac{\partial F}{\partial n_y}\frac{\partial F}{\partial m_y} + \frac{1}{4}\frac{\partial F}{\partial n_{xy}}\frac{\partial F}{\partial m_{xy}}\right),$$

$$P_\kappa = 16\lambda_1^2\left(\left(\frac{\partial F}{\partial m_x}\right)^2 + \frac{\partial F}{\partial m_x}\frac{\partial F}{\partial m_y} + \left(\frac{\partial F}{\partial m_y}\right)^2 + \frac{1}{4}\left(\frac{\partial F}{\partial m_{xy}}\right)^2\right). \tag{33}$$

If these equations are substituted into eqns (32), and the result into eqn (31), the λ_1 term cancels from the numerator and the denominator, to give

$$\boldsymbol{\sigma} = \frac{1}{3\lambda(z)}\mathbf{A}\left[\frac{\partial \mathrm{F}}{\partial \mathbf{n}} + 4\mathrm{z}\frac{\partial F}{\partial \mathbf{m}}\right], \tag{34}$$

where

$$\lambda(z) = \frac{1}{\sqrt{3}}\sqrt{F_1 + 2F_2 z + F_3 z^2} \tag{35}$$

in which

$$F_1 = \left(\frac{\partial F}{\partial n_x}\right)^2 + \frac{\partial F}{\partial n_x}\frac{\partial F}{\partial n_y} + \left(\frac{\partial F}{\partial n_y}\right)^2 + \frac{1}{4}\left(\frac{\partial F}{\partial n_{xy}}\right)^2,$$

$$F_2 = 4\left(\frac{\partial F}{\partial n_x}\frac{\partial F}{\partial m_y} + \frac{1}{2}\left(\frac{\partial F}{\partial n_x}\frac{\partial F}{\partial m_y} + \frac{\partial F}{\partial n_y}\frac{\partial F}{\partial m_x}\right) + \frac{\partial F}{\partial n_y}\frac{\partial F}{\partial m_y} + \frac{1}{4}\frac{\partial F}{\partial n_{xy}}\frac{\partial F}{\partial m_{xy}}\right),$$

$$F_3 = 16\left(\left(\frac{\partial F}{\partial m_x}\right)^2 + \frac{\partial F}{\partial m_x}\frac{\partial F}{\partial m_y} + \left(\frac{\partial F}{\partial m_y}\right)^2 + \frac{1}{4}\left(\frac{\partial F}{\partial m_{xy}}\right)^2\right), \tag{36}$$

from which the stress at any level can be uniquely determined. Now,

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \mathbf{A}^{-1}\boldsymbol{\sigma}, \tag{37}$$

which allows a simple relationship to be written between the von Mises' normals at each level and the Ilyushin stress resultant normals:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{1}{3\lambda(z)}\left(\frac{\partial F}{\partial \mathbf{n}} + 4z\frac{\partial F}{\partial \mathbf{m}}\right). \tag{38}$$

Now, consider some finite strain resultant increment ($\mathrm{d}\boldsymbol{\varepsilon}$, $\mathrm{d}\boldsymbol{\kappa}$), being applied in the above stress state, such that no unloading from yield takes place at any level in the plate. If the direction of $\mathrm{d}\boldsymbol{\varepsilon}$ and $\mathrm{d}\boldsymbol{\kappa}$ does not coincide with the surface normal, then a change of stress resultants **dn** and **dm** must take place such that the revised resultants remain on the yield surface.

At each level z, the stress state corresponding to the exact Ilyushin yield surface is given by eqn (34). It is easily verified that

$$\mathbf{E}^* \frac{\partial f}{\partial \boldsymbol{\sigma}} = 0, \tag{39}$$

so that if **de** and $\partial f/\partial \boldsymbol{\sigma}$ are not coincident then a change in stress will occur. Using (22), the expression for $\mathbf{E}^*$ may be written:

$$\mathbf{E}^* = \mathbf{E} - \frac{16\mathbf{E}_m z^2 + 8\mathbf{E}_{mn} z + \mathbf{E}_n}{R(16z^2 + 8\psi z + \omega)}, \tag{40}$$

where

$$\begin{aligned} \mathbf{E}_m &= \mathbf{E}\frac{\partial F}{\partial \mathbf{m}}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}} E, \\ 2\mathbf{E}_{mn} &= \mathbf{E}\frac{\partial F}{\partial \mathbf{m}}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}}\mathbf{E} + \mathbf{E}\frac{\partial F}{\partial \mathbf{n}}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}}\mathbf{E}, \\ \mathbf{E}_n &= \mathbf{E}\frac{\partial F}{\partial \mathbf{n}}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}}\mathbf{E}, \end{aligned} \tag{41}$$

and

$$\begin{aligned} R &= \frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}}\mathbf{E}\frac{\partial F}{\partial \mathbf{m}}, \\ \psi &= \frac{1}{R}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{m}}\mathbf{E}\frac{\partial F}{\partial \mathbf{n}} = \frac{1}{R}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}}\mathbf{E}\frac{\partial F}{\partial \mathbf{m}}, \\ \omega &= \frac{1}{R}\frac{\partial F^{\mathrm{T}}}{\partial \mathbf{n}}\mathbf{E}\frac{\partial F}{\partial \mathbf{n}}. \end{aligned} \tag{42}$$

Equation (40) is of a form which may readily be integrated, so that the elasto-plastic rigidities are

$$\begin{aligned} \mathbf{C}^* &= \mathbf{E} - \frac{1}{R}(L_2\mathbf{E}_m + 2L_1\mathbf{E}_{mn} + L_0\mathbf{E}_n), \\ \mathbf{B}^* &= -\frac{1}{R}(L_3\mathbf{E}_m + 2L_2\mathbf{E}_{mn} + L_1\mathbf{E}_n), \\ \mathbf{D}^* &= \frac{4}{3}\mathbf{E} - \frac{1}{R}(L_4\mathbf{E}_m + 2L_3\mathbf{E}_{mn} + L_2\mathbf{E}_n), \end{aligned} \tag{43}$$

where the constants L_i are given by

$$L_i = \int_{-1/2}^{1/2} \frac{(4z)^i}{16z^2 + 8\psi z + \omega}\,\mathrm{d}z, \tag{44}$$

whence

$$
\begin{aligned}
L_0 &= \frac{1}{4\sqrt{\omega-\psi^2}}\left[\tan^{-1}\left(\frac{2+\psi}{\sqrt{\omega-\psi^2}}\right)+\tan^{-1}\left(\frac{2-\psi}{\sqrt{\omega-\psi^2}}\right)\right], \\
L_1 &= \frac{1}{8}\log_e\left|\frac{4+4\psi+\omega}{4-4\psi+\omega}\right|-\psi L_0, \\
L_2 &= 1-\omega L_0-2\psi L_1, \\
L_3 &= -\omega L_1-2\psi L_2, \\
L_4 &= \frac{4}{3}-\omega L_2-2\psi L_3.
\end{aligned} \tag{45}
$$

These integrals may be evaluated for the cases where $\omega \geqslant \psi^2$, which can be shown to be identical to the condition imposed by the Schwarz inequality (6), and so is always satisfied.

The case of $\omega = \psi^2$ corresponds to the boundary $Q_t Q_m = Q_{tm}^2$. It may be shown that $\mathbf{E}^*$ is independent of z, which is associated with the fact that $\boldsymbol{\sigma}$ is constant through the depth for this case. The elasto-plastic rigidities are then simply

$$
\mathbf{C}^* = \mathbf{E}^*, \quad \mathbf{B}^* = 0, \quad \mathbf{D}^* = \tfrac{4}{3}\mathbf{E}^*. \tag{46}
$$

The above approach is exact for a point on the Ilyushin yield surface subject to an arbitrary but finite strain resultant increment. Comparisons with the results obtained from multi-layer analyses show that this method gives the correct rigidities, whereas the traditional analysis gives rigidities which differ significantly.

NUMERICAL EXAMPLE

As an example of the use of the equations and theory described here, and to illustrate the techniques described in the earlier paper (Burgoyne and Brennan, 1993), consider an example. Suppose that the non-dimensional stress resultants are known as a result of some other calculation. In this case, take:

$$
\mathbf{n} = (0.20, 0.10, 0.04), \quad \mathbf{m} = (-0.01, -0.02, 0.01).
$$

These values are clearly dominated by the in-plane loading, and relate to a point close to the singular point on the yield surface. This will be reflected several times in the subsequent calculations, where convergence will not be as rapid as it would be for a general point on the surface. The corresponding quadratic stress intensities are:

$$
(Q_1, Q_2, Q_{12}) = (0.0348, 0.0006, -0.0003).
$$

The position on the exact Ilyushin yield surface, ($Q_t = \eta Q_1$, $Q_m = \eta Q_2$, $Q_{tm} = \eta Q_{12}$) and the corresponding parameters α, β and γ are sought. The initial estimate of η is derived from Ivanov's approximate yield surface (Robinson, 1971)

$$
\eta_0 = 28.2676
$$

and the other starting values for the iteration can be found from the normal direction of the surface:

$$
(\alpha_0, \beta_0, \gamma_0) = (0.46127, 0.66930, 0.01331).
$$

These values are used as the starting point for an iterative procedure. In this case, six iterative steps are needed for convergence, which is more than usual and reflects the fact that the point in question is near the discontinuity in slope of the surface. The final values for α, β and γ are

$$(\alpha, \beta, \gamma) = (0.389085134, 0.616994746, 0.008402618)$$

and the corresponding value of η is 28.25238.

The final values of the quadratic stress resultants are then

$$Q_t = 0.983182741,$$
$$Q_{tm} = -0.008475714,$$
$$Q_m = 0.016951427.$$

Two approximate yield surfaces have been quite widely used, referred to as the approximate Ilyushin yield surface and the Ivanov yield surface (Robinson, 1971). Substitution of the above values into the approximate Ilyushin yield surface, gives a value of 1.00503 and in the Ivanov yield surface gives 0.99946. Both of these quantities are close to unity, which indicates that the *value* of the approximating function is locally quite good; other starting stress resultants would give points which differed much more significantly between the various yield surfaces. The normal direction to the yield surface is also of interest however, Table 1 shows both the position of the appropriate point on the yield surface, and also the unit normal direction to the yield surface at that point, for all three surfaces, expressed in the three-dimensional space of the quadratic stress resultants. The angular differences between the normal directions of the approximate surfaces and the normal to the exact surface are also shown.

Table 1. Comparisons between the yield surfaces in quadratic stress resultant space

	Position on yield surface			Unit normal to yield surface		
	Exact Ilyushin	Ivanov	Approx. Ilyushin	Exact Ilyushin	Ivanov	Approx. Ilyushin
Q_t	0.983183	0.983714	0.978264	0.711826	0.730224	0.654654
Q_{tm}	−0.008476	−0.008480	−0.008433	−0.554576	−0.519496	−0.377964
Q_m	0.016951	0.016961	0.016867	0.430987	0.443731	0.654654
Difference between normal directions (radians)					0.04160	0.29170

Similarly, both the position and normal direction of the appropriate point on the six-dimensional yield surface in stress resultant space can also be found, as shown in Table 2.

Table 2. Comparisons between the yield surfaces in stress resultant space

	Position on yield surface			Unit normal to yield surface		
	Exact Ilyushin	Ivanov	Approx. Ilyushin	Exact Ilyushin	Ivanov	Approx. Ilyushin
n_x	1.063059	1.063346	1.060397	0.763681	0.769390	0.779425
n_{xy}	0.212612	0.212669	0.212079	0.551447	0.560776	0.578540
n_y	0.531530	0.531673	0.530199	0.029749	0.027368	0.022500
m_x	−0.053153	−0.053167	−0.053020	−0.297488	−0.273680	−0.225001
m_{xy}	0.053153	0.053167	0.053020	−0.145514	−0.125437	−0.024115
m_y	−0.106306	−0.106335	−0.106040	−0.046238	−0.046753	−0.077943
Difference between normal directions (radians)					0.03311	0.14857

The Tangential Rigidity Matrix [as in eqn (7)] can then be determined by both the present theory and the traditional theory, using the Q_i values determined above from the Exact Ilyushin Yield Surface.

By the present theory [eqns (42)–(44)], the rigidity matrix becomes:

$$\begin{bmatrix} +0.1917 & +0.0513 & -0.2209 & +0.0891 & +0.0935 & -0.0326 \\ & +0.9175 & -0.0629 & +0.0935 & +0.0428 & +0.0135 \\ & & +0.2852 & -0.0326 & +0.0135 & -0.0292 \\ & & & +0.3088 & +0.1122 & -0.3037 \\ \text{(Symmetric)} & & & & +1.2366 & -0.0716 \\ & & & & & +0.3648 \end{bmatrix},$$

while by the traditional theory [eqn (14)], the matrix is:

$$\begin{bmatrix} +0.2943 & +0.0636 & -0.1763 & +0.3792 & +0.1650 & +0.0620 \\ & +0.9208 & -0.0590 & +0.1270 & +0.0552 & +0.0208 \\ & & +0.3060 & +0.0947 & +0.0412 & +0.0155 \\ & & & +1.1296 & +0.3113 & -0.0333 \\ \text{(Symmetric)} & & & & +1.2948 & -0.0145 \\ & & & & & +0.4612 \end{bmatrix}.$$

Both matrices are indeterminate, as would be expected, but the matrix derived from the present analysis satisfies the more stringent conditions given earlier; it is also the matrix to which a multi-layer analysis would converge for large strains. Similar matrices can be derived by both methods from the Ivanov and Approximate Ilyushin Yield Surfaces, and show further differences from the correct version derived here.

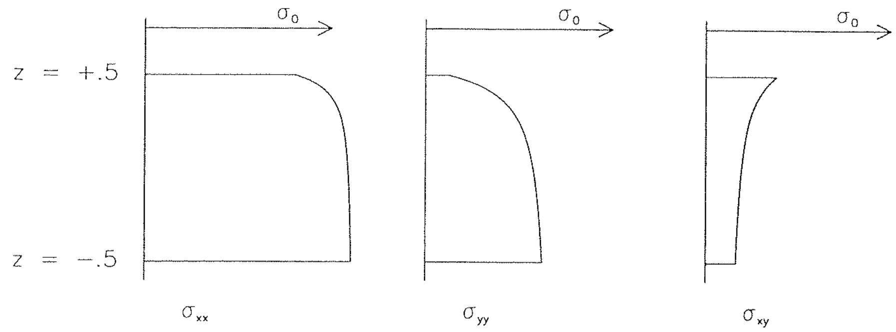

Fig. 2. Stress variation through thickness for **n** = (0.20, 0.10, 0.04), **m** = (−0.01, −0.02, 0.01).

Finally, the stresses throughout the thickness of the plate can be determined from eqns (33)–(35). These are shown in Fig. 2. Although there is no stress reversal in this case (since in-plane effects are dominant), there are significant changes in stress through the thickness of the plate.

CONCLUSIONS

It has been shown that the traditional method of deriving the elasto-plastic rigidity matrix in stress resultant space is in error, because the assumptions made about the relative magnitudes of elastic and plastic strain components do not hold in stress resultant space. However, it has also been shown that by making use of the normal direction to the exact Ilyushin yield surface, the strain distribution through the plate can be determined, from which the stress distribution can be found from von Mises' yield condition. This knowledge then allows the elasto-plastic rigidities to be calculated, as though a multi-layer analysis in terms of stresses had been undertaken.

A numerical example has been carried out to demonstrate the numerical procedures described, and to show the significant differences between the two theories.

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