INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 29, 105–121 (1990)

NUMERICAL INTEGRATION STRATEGY FOR PLATES AND SHELLS

C. J. BURGOYNE

Engineering Department, Cambridge University, Trumpington St, Cambridge, CB2 1PZ, U.K.

M. A. CRISFIELD

Dept of Aeronautics, Imperial College, Prince Consort Road, London, SW7 2BY, UK

SUMMARY

The paper compares the overall performance of a wide range of numerical procedures that can be used to integrate through the thickness of plates and shells. Results are presented for the accuracy of the calculations when there are discontinuities in the stress through the depth of the plate, and the available methods are ranked according to their accuracy.

INTRODUCTION

The present work developed from an earlier study,[1] which indicated that in certain circumstances it might be preferable to use a simpler integration strategy (such as the trapezium rule) rather than a more complex one (such as higher order Gauss quadrature). It is based on the assumption that the stress distribution is non-linear through the thickness of the plate. This non-linearity can be caused by cracking, in the case of concrete slabs, or yielding, in the case of steel plates. In both cases there will be discontinuities in the function to be integrated, and/or its derivative through the thickness. In each case, the form of the stress variation is known from the constitutive relations for the material, but the positions of the discontinuities are not; they vary both over the area of the plate and as the loading progresses.

All the integration strategies available consist, in essence, of taking the algebraic sum of the value of the function to be integrated (y_i) at a number of specified abscissae (x_i), multiplied by weighting factors (a_i). The differences between the techniques lie in the sophistication of the assumptions made in deriving the weighting factor, and/or the positions of the integration points.

In plate and shell problems, the stresses to be integrated are themselves found as the result of complex calculations, part of which depends on the stress history at any particular point. Thus, it would be difficult to vary the positions of the integration points (x_i) through the course of the calculation, since this would mean that the historical information would no longer be directly available. The requirement is thus for an integration method that is as accurate as possible, consistent with the need to fix these integration points before the calculation starts.

Some workers[2,3] have attempted to trace the elasto-plastic[2] or crack[3] interface, but most analysts have used fixed integration stations which fall into the category of schemes covered in this paper. The simplest of these schemes involves dividing the depth into even layers and lumping the properties at their centroids. This procedure (henceforth called the 'centroidal' rule) was used in early work by Popov *et al.*[4] and Backlund and Wennestrom[5] and continues to find favour.[6–8] Between seven[7] and sixteen[8] layers appear to be used.

0029–5981/90/010105–17$08.50

Received 22 June 1988
Revised 5 May 1989

Some early workers[9,10] investigated more sophisticated integration procedures. Marcal and Pilgrim[9] used eleven trapezoidal integration stations while Stricklin *et al.*[10] found that Simpson's rule was more accurate. The latter authors rejected Gaussian integration because it did not directly account for yielding on the surface as soon as it occurred. Crisfield[11] argued that 'this restriction is unlikely to be of major importance from the point of view of predicting collapse loads' and adopted Gaussian integration with five stations through the depth.

Cormeau[12] also advocated Gaussian integration (using six stations) and clearly demonstrated its advantages in comparison with the centroidal rule. Irons[13] noted that the Gaussian rules 'are quite competitive even when there are discontinuities' and showed that, for such a problem, 'an 8-point Gaussian rule completely outperforms a 16-point Newton–Cotes rule'. Crisfield[14] has used Gaussian integration for the analysis of masonry arches involving a no-tension material. In order to reproduce 'mechanism solutions' with hinges effectively on the surface, at least eight Gaussian stations were required. Many more points would have been needed if the centroidal rule had been adopted.

With a view to the immediate incorporation of yielding on the surface, Cormeau[12] investigated the Lobatto rule.[15] Using the inelastic deflections for a biquadratic shell as a basis for comparison, Cormeau observed that, in comparison with the standard Gaussian procedure, 'the numerical results show that this feature does not improve the accuracy'. In relation to the analysis of concrete, Bergan[16] advocates the Lobatto rule because of the sampling points on the surface. He notes that, under pure bending, the outer fibre will exceed the cracking strain by 17 per cent by the time the outer integration point of a 7-point centroidal rule has registered cracking, and by 5·4 per cent for the equivalent Gauss rule.

The present authors would contend that the latter error is relatively unimportant. It is of more significance that the rigidities to be used in subsequent analyses, which are determined from integrals of the stresses through the thickness, must be as accurate as possible. This will enable the post-cracking (or post-yield) response to be followed more accurately, even if the precise load at which the first crack formed is missed. The field in which the accurate detection of first yielding is most likely to be important is that of the buckling behaviour of thin shells. Even here, it is the accurate representation of the reduced stiffness in the early post-yielding period, rather than the actual detection of surface yielding, that will matter. It is certainly possible that a reasonably high order Gaussian rule will out-perform the Lobatto rule even though the latter has points on the surface. A further advantage of Gaussian integration is that it gives exact solutions with only two points when the response is completely elastic, for both the moment resultant and the force resultant.

Further studies are in progress to investigate the relative importance of the detection of first yield (or cracking) and the accuracy of the subsequent calculations of stiffness; these will be reported in due course. In the present paper, a measure of the overall accuracy of an integration strategy will be determined, which permits comparisons between methods on a rational basis. A variety of integration techniques will be built up from standard formulae, and applied to a range of idealized problems. The accuracy of the various techniques will be compared, and recommendations made regarding the best technique to use in different circumstances.

The results obtained in this paper are more generally applicable to the numerical integration of discontinuous functions. In plate and shell problems, the range of integration is normally fixed (to the thickness of the plate), whereas in the general problem, the range of integration may vary. This is not insignificant; for some of the integration techniques an interpolating function can be identified which can itself be integrated analytically over any range. It is then possible to find revised quadrature formulae for any integration range using invariant integration points, which will be of considerable benefit when the function is known only at certain points, or is difficult to calculate.

Detailed results are not presented here for these problems, but integration techniques which give accurate results, and which are based on interpolation methods, will be identified.

METHODS AVAILABLE FOR NUMERICAL INTEGRATION

A variety of techniques can be used for numerical integration. At their simplest, most rely on fitting an approximate function through a number of points at which the integrand is calculated, and then integrating analytically the approximating function. A rigorous mathematical derivation of each method will not be given here, but the principles of each method will be described, and references given to more detailed explanations. Where available, references will be provided to accurately tabled values of numerical constants. A good summary of many of the available methods is given in Reference 17, while more mathematical treatments are given in References 15 and 18.

All the methods presented assume that the function to be integrated varies smoothly within the integration range, an assumption that is not valid in our case. However, it is not usually feasible to take full account of the presence of the discontinuities, and it was the need to determine the accuracy of the methods, when faced with problems in which the discontinuities cannot be avoided, that led to the work described in this paper.

Gauss method

Gaussian quadrature is one of the oldest and best known integration techniques, but can be used effectively only when the range of integration is constant. In essence the method works by assuming that the function can be evaluated at n points, whose positions are not yet fixed. There will thus be $2n$ unknowns to be evaluated (n positions x_i and n weighting functions a_i). These unknowns will be determined on the assumption that the function y is a polynomial of degree $2n-1$ and that the integral over a specified region is to be exactly given by the summation $\sum a_i y_i$. Gaussian integration is thus the most general of the methods that will be considered, since none of the parameters are fixed.

The evaluation of these unknown coefficients involves the solution of non-linear simultaneous equations, which are dependent on the range of integration required. For this reason, results are usually quoted for integration in the fixed range -1 to $+1$. Values of the constants for all values of n between 2 and 16, quoted to 15 significant figures, are given in Reference 19, while some other solutions for values of n between 16 and 96 are given in Reference 20, quoted to 20 significant figures.

Lobatto's method

It may, arguably, be desirable to ensure that there are integration points at the limits of the integration. Gaussian integration does not provide such points, but this problem can be overcome, and most of the accuracy retained, by the use of Lobatto's method[20] (which should, more accurately, be attributed to Radau[21]). The parameters are derived in a similar way to those of Gauss, with the difference that the first and last abscissae are fixed at -1 and $+1$ respectively. Two degrees of freedom are thus removed from the calculation, so it is now only possible to find an exact solution for polynomials of degree $2n-3$. Values of the weights and abscissae are given in Reference 20 for Lobatto methods with up to 10 points, although most of the results are quoted only to seven significant figures; Reference 18 gives, additionally, the 11-point form.

Equal weight methods

There are some occasions when it is desirable to have equal weights associated with each abscissa. In particular, if the calculation (or measurement) of the integrands is subject to significant error, it is preferable that errors in one term do not have undue influence on the overall calculation. Chebyshev derived expressions for such forms of numerical integration, in which the unknowns are the positions of the n abscissae. Unlike the Gaussian form, there are n unkowns, rather than $2n$, so it is possible only to find exact solutions for polynomials up to degree $n-1$.

For integration in the range -1 to $+1$, the abscissae can be shown to be the roots of the Chebyshev quadrature polynomials. For values of n between 1 and 7, and also for $n=9$, all the roots are real, but for $n=8$ and $n>10$, complex roots exist, so no formulae can be found for these problems.[22] Nevertheless, if it is desirable to use more points, repeated use can be made of the simpler formulae.

Cotes' method

Cotes' method is a generalization of the two traditional methods of numerical integration; the trapezium rule and Simpson's rule. If the value of a function is known at n points, it is possible to fit a polynomial of degree $n-1$ through those points; that polynomial can then be integrated to give the required solution. If the function is known at equally spaced points, and the integration is carried out between the first and the last points, the method is known as Cotes' method.

Cotes' method is often expressed by the formula

$$\int_{x_0}^{x_n} y(x)\cdot \mathrm{d}x=(x_n-x_0)\sum_{k=0}^{n} C_k^n\cdot y_k$$

and the coefficients C_k^n are tabulated in the form $N\cdot C_k^n$, where N is chosen to give integral values. Coefficients in this, or similar, form are given in Reference 17 for up to 7 points, and in Reference 23 for up to 11 points; Reference 18 gives results for up to 20 points.

The error terms associated with the Cotes formulae are such that there is usually not much advantage in using the even order forms,[17] so these are rarely used. The higher order formulae (9-point, 11-point and upwards), involve some negative weighting coefficients.[23] This is symptomatic of the fact that attempts to fit a high order polynomial through a limited number of points often leads to spurious oscillations in the approximating function, which significantly increases the error of the associated integral.

To avoid problems with higher order formulae, lower order Cotes formulae are often applied repeatedly over adjacent regions. The end points of each region overlap, so the values of the function at these points are used twice. Thus, the '5-point Cotes rule applied twice' will make use of 9 distinct integration points; this rule passes two separate quartic polynomials through each half of the region. The result is not the same integration rule as the 9-point Cotes formula, which passes an 8th order polynomial through all 9 points. Figure 1 shows a variety of approximating functions for 9-point Cotes integration.

Romberg's method

Romberg's method is a modification of the Cotes method. If the integrand is assumed to be differentiable, to whatever degree is needed, then the error in the trapezium rule is expressible as a power series in the interval size. By combining the result for one interval size with the result for twice as many intervals, it is possible to eliminate the highest error term. This process can be repeated as many times as necessary, each time doubling the number of intervals, while eliminating a further error term, and so increasing the accuracy of the calculation. Romberg's method is outlined in Reference 17, and explained in detail in References 24 and 25.

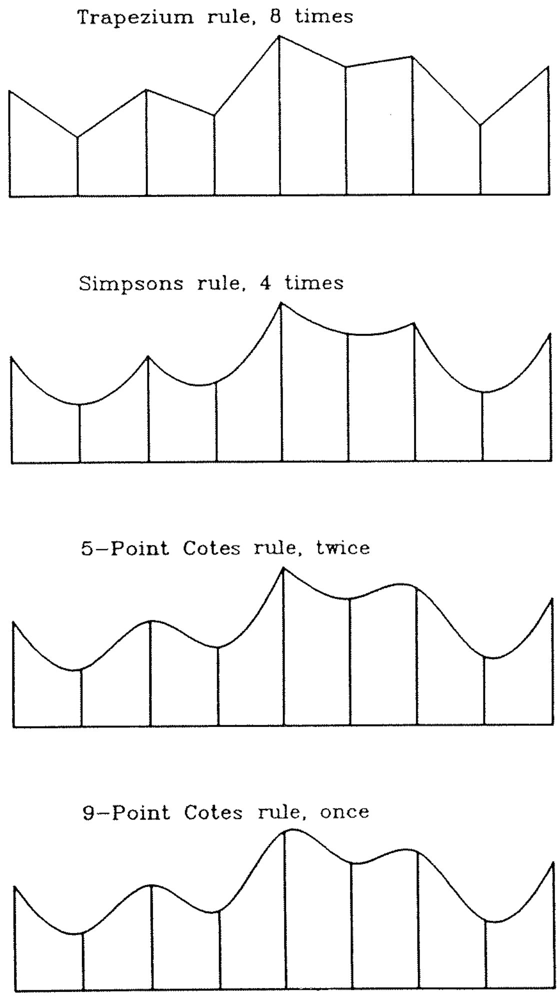

Figure 1. Alternative interpolating functions for 9-point integration

The first stage of this process yields a formula identical to Simpson's rule, while the second stage yields a result identical to the 5-point Cotes rule (i.e. 4 intervals), so no new formulae are obtained so far. However, repeated applications of the method do yield new formulae. The third stage of the process yields an 8-interval (9-point method), which will be considered below. This can be regarded as a modification of the 5-point Cotes rule used twice, and, unlike the 9-point Cotes rule, does not have negative coefficients. Other formulae, using 16, 32, 64 etc. intervals, can be derived easily, but they will not be considered further here.

Analytical trapezium method

A modification of the trapezium method can be used for integration to find the moment within a plate. The force in the plate is found by the trapezium rule in the normal way, which involves an integration of a series of linear interpolations of the stresses between the integration points. To calculate the moment, the same interpolated functions are used for the stresses, with the moment arm specified exactly. The integration between each pair of integration points is thus applied to a quadratic, rather than a linear function, so higher accuracy can be expected by this means.

It is not possible to derive simple expressions of the form $M=\sum a_i y_i$, in which y_i is the integrand $x_i\sigma_i$, which includes the lever arm component x_i. Instead, expressions of the form $M=\sum a_i x_i \sigma_i =\sum b_i \sigma_i$ can be derived for each particular problem, in which the lever arm component x_i is included in the weighting function b_i.

Bates' method

A method proposed by Bates[26] bears some similarity to the analytical technique described above, but can be applied to almost any function. It is claimed that the method yields improved results for discontinuous functions (when compared with the trapezium rule), and is still accurate for smooth functions.

The basic assumption is that the integrand (φ) can be factorized into 2 functions f and g.

$$I=\int_a^b \varphi(x)\cdot \mathrm{d}x=\int_a^b f(x)\cdot g(x)\cdot \mathrm{d}x$$

where $f(x)$ is a piecewise or continuous function of order one, and $g(x)$ is a continuous function. The integral is now evaluated using Simpsons's rule, with $2n$ intervals ($2n+1$ points). Alternate values for both f and g are then eliminated by taking the average values of the adjacent points (implicitly assuming that both f and g are piecewise linear), which yields, after some manipulation

$$I=\frac{(b-a)}{n}\{\tfrac{1}{2}f(0)[g(0)+\Delta g_1]+f(1)g(1)+\ \dots\ +f(n-1)g(n-1)+\tfrac{1}{2}f(n)[g(n)-\Delta g_n]\}$$

where $\Delta g_i=[g(i)-g(i-1)]/3$.

It is intended by Bates that the function $g(x)$ should be the moment arm when calculating integrals for moments, but in fact the method is more generally useful. The expression for the integral can be expanded, and f and g recombined where possible, to give

$$I=\frac{(b-a)}{n}\{\varphi(0)/3+\varphi(1)+\dots+\varphi(n-1)+\varphi(n)/3\}+\frac{(b-a)}{6n}\{f(0)g(1)-f(n)g(n-1)\}$$

from which it can be seen that the method is similar to the trapezium rule, with two additional terms and modified weighting for the end points. Note that the method is not symmetrical with respect to f and g; the functions cannot be interchanged.

If it is assumed that $g(x)=x$, then the result is identical to the analytical technique. However, an interesting result is obtained if $f(x)$ is set to unity. The additional terms can then be rewritten in terms of the unfactored integrand (φ), to give

$$I=\frac{(b-a)}{6n}\{2\varphi_0+7\varphi_1+6\varphi_2+6\varphi_3+\ \dots\ +6\varphi_{n-2}+7\varphi_{n-1}+2\varphi_n\}$$

This can be visualized as an end correction to the trapezium rule. Detailed results will be presented below for this version of Bates' method.

Centroidal method

A technique that is widely used in integration through the thickness of a plate or slab is the centroidal rule. In this method, the plate is divided into a number of layers, of equal thickness. The integrand is assumed to be constant over each region, so an equal weight, equal interval method is obtained which is extremely easy to program. The method is alternatively known as the mid-point or rectangular rule.

Spline curve

Spline curves are widely used in computer graphics applications[27] to pass a smooth curve through a number of points. A number of different versions of these techniques are available, including tensioned splines,[28] to improve the 'fairness' of the curves, and parabolic blending, which is used to localize the effect of a variation at one node. Since these methods are, in essence, attempts to find a good approximation to a function, it seems only logical to extend the process and integrate the approximating function thus obtained. A thorough review of the use of spline curves in this way is given in Reference 29.

Only the basic, untensioned spline will be considered here, and for simplicity it will be assumed that the integration points are equally spaced over the whole region to be integrated. Between each pair of abscissae, the integrand is assumed to be a cubic polynomial, and can thus be expressed in terms of 4 parameters. These parameters can be determined by specifying the value of the function at each node (these are the independent variables of the problem), and ensuring continuity of slope and curvature of the function between adjacent regions. Taken together, these conditions yield sufficient equations to specify all but two of the parameters, whatever the number of regions.

The last two conditions can be found by specifying the slope or curvature at the ends of of the integration region. In this paper, two methods will be used; the first assuming that the curvature at the ends can be found by extrapolating the curvature at the two adjacent internal abscissae [Spline], while the second assumes that the curvature at the ends is zero [Spline-0].

The determination of the interpolating function involves a matrix inversion, after which the weighting functions can be determined easily. For regular node spacings with integration over the full range, it is possible to produce tables of abscissae and weights as for the other methods. The primary benefit of the method, however, is the ease with which it can be extended to irregular node spacings and integrations over other ranges.

COMPARISON OF METHODS

Figure 2 shows diagrammatically the distribution of the abscissae and the associated weighting functions for fourteen different methods of integration using 9 integration points. In each case the vertical line represents the range of integration, while the horizontal lines represent the weights associated with each point; the longer the lines, the greater the weight. The vertical position of the lines represents the relative positions of the abscissae within the integration range. All the charts are shown to the same scale, with the average value of the weights shown for clarity.

The weights for the two spline methods (Figure 2) differ significantly only at the ends; in the centre of the integration region, the weights of the various abscissae are nearly equal. This is not particularly surprising, since changing the value of the function at one of the internal nodes will only affect the spline curve locally, and similar changes will result whichever node is altered. Thus, the weights of all nodes will, in the absence of end effects, be equal. Thus, apart from the end variations, the spline curve expressions are closest to those of the trapezium rule.

The 9-point version of Simpson's rule, the 5-point Cotes formula used twice and the Romberg method are seen to differ little from one another. However, the strange behaviour of the 9-point Cotes formula is obvious, and it is clear that big discrepancies in solutions can be expected as discontinuities pass across the abscissae.

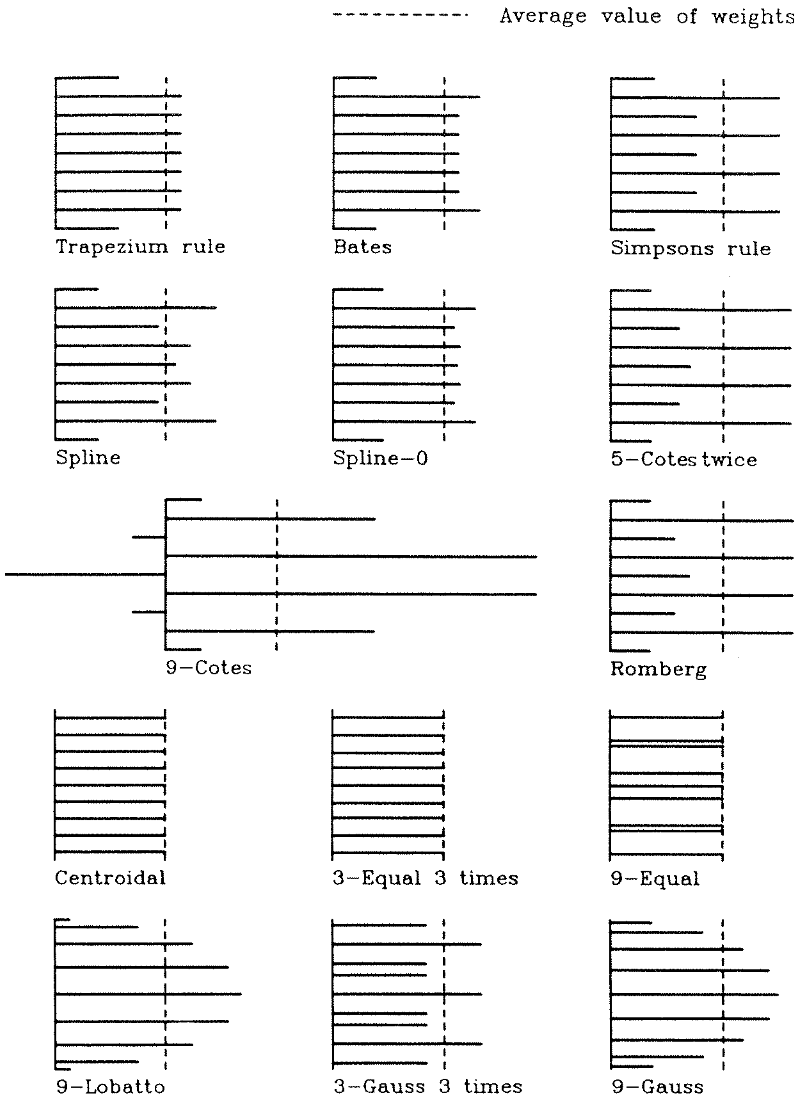

Figure 2. Weights and abscissae for 9-point integration

The uneven distribution of the abscissae in the 9-point equal weight method is clear, and compares with the even distribution of the 3-point equal weight method used 3 times, which is almost indistinguishable from the centroidal rule at this scale.

The Gaussian integration techniques have a much smoother distribution of weights and abscissae than the other methods. The abscissae are closer together at the edge of the integration range, but with low weights, whereas at the centre, there are fewer points but the weights are increased. The low weight associated with the extreme points in the Lobatto's method can also be seen.

ORDERS OF INTEGRATION USED

For the present study, all reasonable integration methods using 13 integration points or less have been studied. The choice of thirteen points as the upper limit was made since this allows the repeated use of several lower order Cotes formulae (3-point 6 times, 4-point 4 times, etc.), and repeated use can be made of several Gaussian integration techniques with twelve points (2-point 6 times, 3-point 4 times, etc.). A total of 130 different integration strategies have been studied. The data for each one were produced as a DATA statement; a BASIC program used each one in turn to produce results for each of the integration problems studied.

TEST CASES CONSIDERED

Four separate integration problems have been studied. These were chosen to represent the worst cases likely to be encountered in a study of plate and shell problems, but the results are likely to be applicable to other problems which involve discontinuities. Concrete, at failure, is often assumed to have a constant stress in the compression zone, but to be cracked in tension, so that no stress is carried. Thus, the stress block is as shown in Figure 3(a). In a structural analysis program, it is necessary to integrate the stress itself, to get the force (Case I), and also the stress multiplied by a lever arm, to get a moment (Case II). This latter problem results in an integrand as shown in Figure 3(b). Both integrands have discontinuities in the value of the function itself, though in the second case the magnitude of the discontinuity varies as the position of the discontinuity varies.

The other two cases are derived from studies of steel plates, in which the outer layers of a plate may be plastic, and so at virtually constant stress, while the centre is elastic. This leads to a stress block without a discontinuity in value, but with a discontinuity in slope. The same two problems are considered to get force (Case III) and moment (Case IV), as shown in Figure 3(c) and 3(d).

Consideration was also given to integrals involving the stress multiplied by the square of a lever arm, since such integrals are used in the calculation of the stiffness of an element, but as will be seen later, the results for Cases I and II are broadly similar, as are the results for Cases III and IV. It was thus decided not to study the additional cases further.

The four cases considered in Figure 3 are only building blocks from which practical situations can be derived. For example, an accurate representation of the behaviour of concrete, which involves compressive yielding and tensile cracking, would incorporate a combination of (a) and (c) (and hence also (b) and (d)). Thus, when considering the results of these studies, it is important not to associate them too closely to concrete or steel. In a real problem a more complicated situation may arise. There may be more than one discontinuity, and the stresses may vary in a more complicated manner away from the discontinuity. Nevertheless, these four problems may be regarded as the basic blocks.

METHOD OF COMPARISON

In each of the problems studied, the position of the discontinuity has been left as a variable (z). The difference between the numerical integration and the exact solution will be the error of the analysis (ε). ε will be a function of z, and to avoid making decisions based on a single value of z, which may be the worst case for one integration method, but the best for another, a method is sought in which the overall accuracy for all values of z is assessed. While such overall accuracy is desirable, it may, in reality, not be the overriding criterion. For example, with plate or shell buckling, the dominant issue may be initial loss of stiffness following first yielding ($z \to 0$ in Figure 3(c)). In contrast, for reproducing plastic mechanism analyses, the dominant situation will involve full 'stress-block' plasticity for the complete section. Yet again, for the rigid plastic

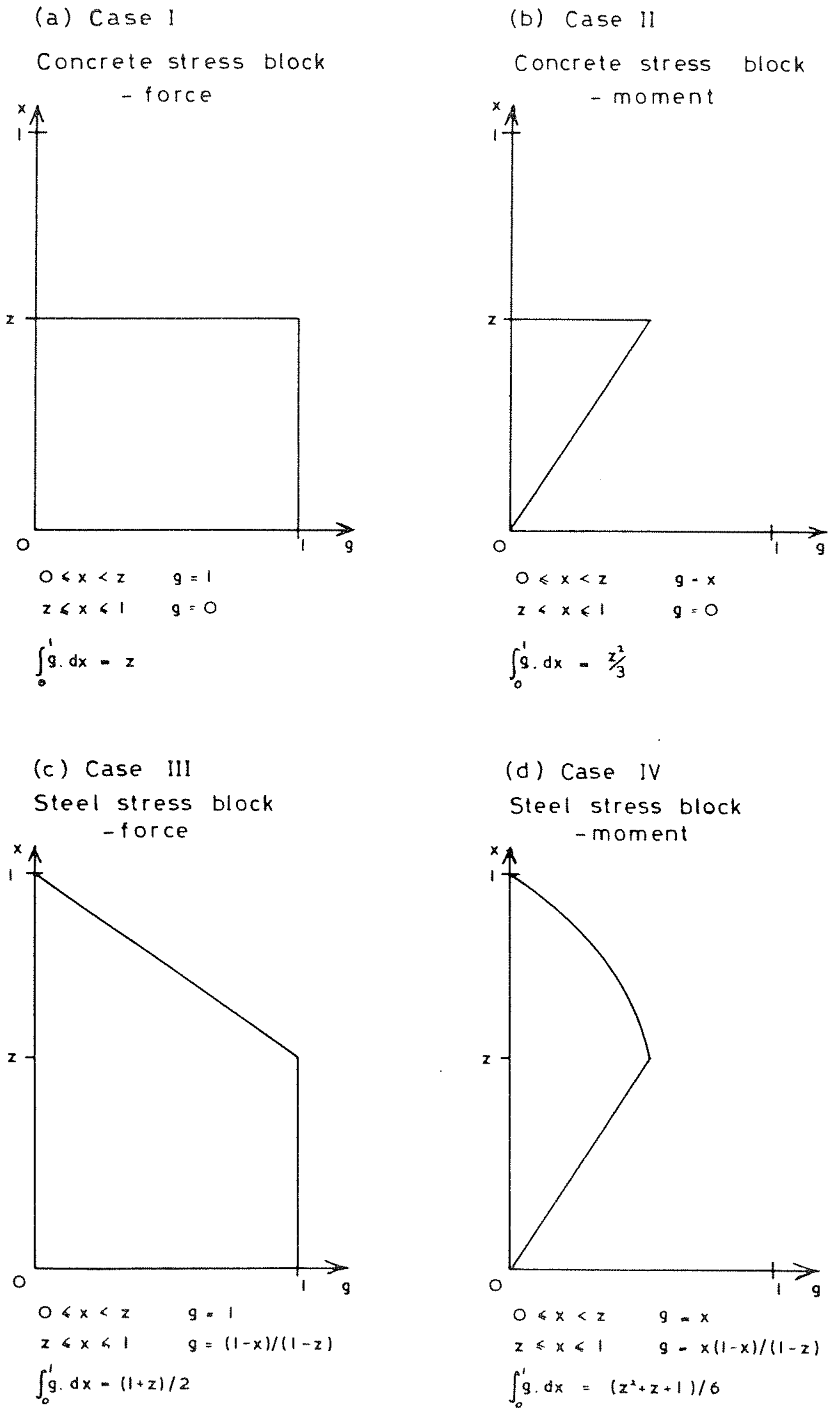

Figure 3. Test cases considered

mechanism analysis of a masonry arch bridge,[14] the most important case would involve a very thin compressive block at the edge. The present work ignores these possible different weightings for the different 'z-positions' and assumes an equal weighting to devise a single measure of overall accuracy.

The method adopted is to take the root mean square error over the whole range of feasible z values. This overall error is itself determined by numerical integration, using Gaussian integration. The error ε will be a discontinuous function of z, but unlike the general problem that is being studied, it is known that these discontinuities will occur at the integration points (Figure 4). It is

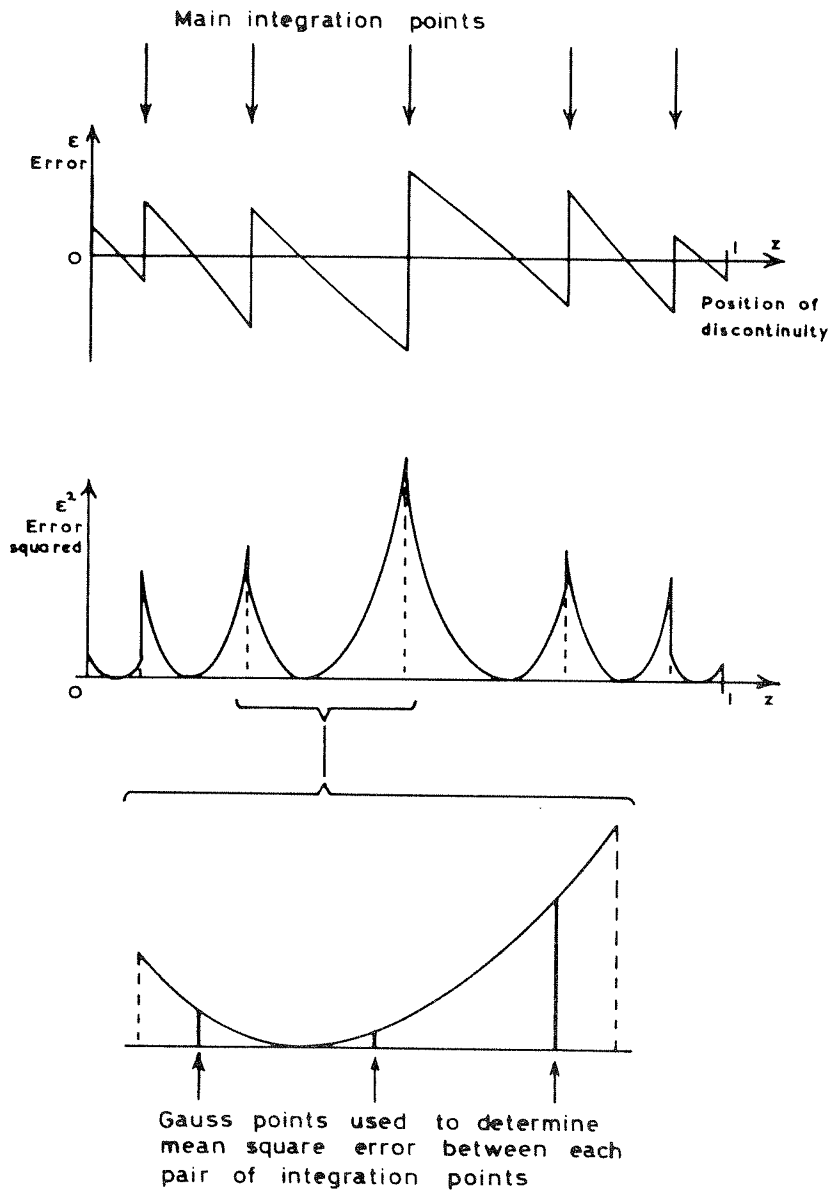

Figure 4. Determination of error for each method

thus possible to use a relatively low order Gaussian integration (3 points) over each region between the main integration points, and still achieve a sufficiently accurate calculation.

COMPARATIVE RESULTS

Tables I (a) and (b) show a summary of the results of the various trials, ranked from the best method to the worst. The first column gives the rank, followed by the type of integration technique used. The third and fourth columns give the number of times that the method is applied and the total number of integration points used. The next four columns (Table I(a)) give the root mean square error in the calculation of the four integrals described above, expressed as a proportion of the error of the best method for that problem. Thus, the most accurate technique will have a value of unity, and all other methods will have higher values. Finally, the last column gives the average of the previous four columns, and it is this value which is used to determine the rank of the method. The first row of the table gives the absolute value of the error for the best method in each of the four cases. Detailed results are given for the first 40 methods (Table I(a)), while for the remaining 90 methods (Table I(b)), only the mean results and the ranks are quoted.

Figure 5 shows the mean error, as shown in the last column of the table, against the number of integration points. For five of the principal methods, lines have been added to make clear the variation of error with number of integration points. (The analytical trapezium results have been omitted from the figure, since they lie almost on top of the trapezium rule results at this scale.)

The table and the figure contain a lot of detail, but some observations can be made.

1. The best method for the problems with discontinuities in the value of the function (Cases I and II) is the centroidal rule applied 13 times. At first sight, this may be surprising, since the centroidal rule approximates the function by a constant over each layer, and so is itself discontinuous over the whole integration range. However, in these problems the integrand is varying linearly over each layer, except for the layer that contains the discontinuity; the contribution to the total integral from all but one of the integration points will thus be exact. For the other methods, the effect of the discontinuity is to propagate errors into the function well away from the discontinuity, with consequential effects on the accuracy.

   The accuracy of the centroidal rule for Cases I and II is due not to the inherent accuracy of the interpolating function, which is very poor, but rather to the cancellation of the errors that arise in these particular problems. However, the centroidal method using 13 points is poor in Cases III and IV, resulting in a final rank of 14th.
2. The 13-point Gauss method is the most accurate technique for Cases III and IV, and has relatively small errors in the other two cases. This results in the best average error, to leave this method with rank 1.
3. The first eight places in the table all go to versions of the Gauss method. The 10-point Gauss method is better than any non-Gauss method considered. For any given number of points, Gaussian integration gives the best overall solution.
4. Lobatto's method and the equal weight methods give similarly good results that are second only to the Gauss methods. Lobatto's method with n points gives results which have similar accuracy to those produced by the Gauss method with $n-2$ points; it has the additional benefit of having abscissae on the surface, and if stresses have to be determined on the surface anyway, very little additional work is required.
5. The best methods using interpolation formulae (Cotes or Spline) are the 13-point spline, 5-point Cotes (3 times) and Simpson's rule (6 times), all of which use 13 points and all of which have very similar errors.

Table I(a)

Rank	Rule	Repeat	No of pts	Relative root mean square errors: Case 1 conc. force	Case 2 conc. moment	Case 3 steel force	Case 4 steel moment	Mean
	Absolute errors of best methods			0·02965	0·01708	0·001367	0·001052	
1	13-Gauss	1	13	1·240	1·180	1·000	1·000	1·105
2	12-Gauss	1	12	1·339	1·274	1·165	1·165	1·236
3	11-Gauss	1	11	1·456	1·385	1·375	1·375	1·397
4	6-Gauss	2	12	1·286	1·273	1·551	1·731	1·460
5	4-Gauss	3	12	1·237	1·233	1·790	2·085	1·586
6	10-Gauss	1	10	1·594	1·516	1·646	1·646	1·601
7	3-Gauss	4	12	1·194	1·192	1·963	2·343	1·673
8	2-Gauss	6	12	1·124	1·124	2·241	2·779	1·817
9	3-Equal	4	12	1·100	1·100	2·295	2·880	1·844
10	4-Equal	3	12	1·160	1·160	2·301	2·807	1·857
11	9-Gauss	1	9	1·761	1·676	2·008	2·010	1·864
12	5-Gauss	2	10	1·519	1·504	2·170	2·425	1·904
13	10-Lobatto	1	10	1·776	1·691	2·221	2·339	2·007
14	Centroidal	13	13	1·000	1·000	2·647	3·467	2·029
15	6-Equal	2	12	1·379	1·373	2·734	3·159	2·161
16	8-Gauss	1	8	1·968	1·873	2·505	2·509	2·214
17	Centroidal	12	12	1·083	1·083	2·983	3·908	2·264
18	5-Equal	2	10	1·334	1·335	2·914	3·595	2·295
19	2-Gauss	5	10	1·348	1·348	2·940	3·625	2·315
20	3-Gauss	3	9	1·591	1·586	3·009	3·520	2·426
21	9-Lobatto	1	9	1·988	1·894	2·830	3·010	2·430
22	Centroidal	11	11	1·181	1·180	3·396	4·452	2·553
23	4-Gauss	2	8	1·855	1·837	3·260	3·652	2·651
24	7-Gauss	1	7	2·230	2·123	3·214	3·222	2·697
25	3-Equal	3	9	1·465	1·465	3·525	4·395	2·713
26	Spline	1	13	1·128	1·139	4·106	5·244	2·904
27	Centroidal	10	10	1·298	1·297	3·915	5·135	2·911
28	5-Cotes	3	13	1·311	1·312	4·059	5·067	2·937
29	Simpson's	6	13	1·230	1·232	4·117	5·208	2·947
30	8-Lobatto	1	8	2·259	2·153	3·736	4·023	3·043
31	Bates'	1	13	1·113	1·122	4·374	5·665	3·069
32	2-Gauss	4	8	1·683	1·683	4·099	5·014	3·120
33	4-Cotes	4	13	1·144	1·145	4·521	5·787	3·149
34	4-Equal	2	8	1·737	1·737	4·206	5·029	3·177
35	7-Cotes	2	13	1·636	1·627	4·394	5·271	3·232
36	Spline-0	1	13	1·099	1·104	4·691	6·062	3·239
37	Spline	1	12	1·234	1·246	4·678	5·968	3·281
38	9-Equal	1	9	1·938	1·923	4·531	4·983	3·344
39	Centroidal	9	9	1·441	1·440	4·580	6·012	3·368
40	6-Gauss	1	6	2·571	2·449	4·278	4·298	3·399

Table I(b)

Rank	Rule	Repeat	No. of pts	Mean error	Rank	Rule	Repeat	No. of pts	Mean error
41	Bates'	1	12	3·466	86	Spline	1	7	7·731
42	Spline-0	1	12	3·664	87	Simpson's	3	7	7·740
43	Spline	1	11	3·752	88	Trapezium	8	9	7·887
44	Simpson's	5	11	3·794	89	2-Gauss	2	4	7·944
45	7-Equal	1	7	3·849	90	4-Equal	1	4	8·013
46	Bates'	1	11	3·960	91	Bates'	1	7	8·095
47	Centroidal	8	8	3·967	92	7-Cotes	1	7	8·189
48	7-Lobatto	1	7	3·985	93	4-Cotes	2	7	8·385
49	3-Gauss	2	6	4·104	94	5-Lobatto	1	5	8·577
50	Spline-0	1	11	4·194	95	Spline-0	1	7	8·701
51	Trap-Anal	12	13	4·239	96	Trap-Anal	7	8	9·079
52	Spline	1	10	4·353	97	Trapezium	7	8	9·581
53	Trapezium	12	13	4·382	98	11-Cotes	1	11	9·774
54	5-Gauss	1	5	4·488	99	6-Cotes	1	6	9·878
55	Bates'	1	10	4·590	100	3-Gauss	1	3	10·053
56	2-Gauss	3	6	4·592	101	Spline	1	6	10·056
57	3-Equal	2	6	4·697	102	Bates'	1	6	10·439
58	4-Cotes	3	10	4·720	103	Centroidal	4	4	10·461
59	Centroidal	7	7	4·777	104	Trap-Anal	6	7	11·281
60	Trap-Anal	11	12	4·794	105	Spline-0	1	6	11·327
61	10-Cotes	1	10	4·870	106	Trapezium	6	7	12·001
62	Spline-0	1	10	4·873	107	3-Equal	1	3	12·151
63	Trapezium	11	12	4·969	108	5-Cotes	1	5	13·408
64	5-Cotes	2	9	5·131	109	Simpson's	2	5	13·684
65	Spline	1	9	5·141	110	Bates'	1	5	14·226
66	Romberg 8	1	9	5·142	111	Trap-Anal	5	6	14·576
67	Simpson's	4	9	5·177	112	4-Lobatto	1	4	15·446
68	6-Equal	1	6	5·295	113	Spline-0	1	5	15·565
69	Bates'	1	9	5·413	114	Trapezium	5	6	15·675
70	Trap-Anal	10	11	5·485	115	Centroidal	3	3	15·677
71	6-Lobatto	1	6	5·569	116	Trap-Anal	4	5	19·913
72	Trapezium	10	11	5·704	117	2-Gauss	1	2	20·488
73	Spline-0	1	9	5·764	118	Bates'	1	4	21·120
74	5-Equal	1	5	5·823	119	Trapezium	4	5	21·758
75	Centroidal	6	6	5·925	120	4-Cotes	1	4	22·609
76	8-Cotes	1	8	6·143	121	Spline-0	1	4	24·140
77	Spline	1	8	6·213	122	Centroidal	2	2	27·713
78	4-Gauss	1	4	6·346	123	Trap-Anal	3	4	29·661
79	Trap-Anal	9	10	6·366	124	Trapezium	3	4	33·250
80	Bates'	1	8	6·526	125	Simpson's	1	3	36·505
81	Trapezium	9	10	6·646	126	Trap-Anal	2	3	51·421
82	Spline-0	1	8	6·977	127	Trapezium	2	3	60·558
83	9-Cotes	1	9	7·289	128	Centroidal	1	1	72·744
84	Trap-Anal	8	9	7·519	129	Trap-Anal	1	2	123·937
85	Centroidal	5	5	7·648	130	Trapezium	1	2	168·698

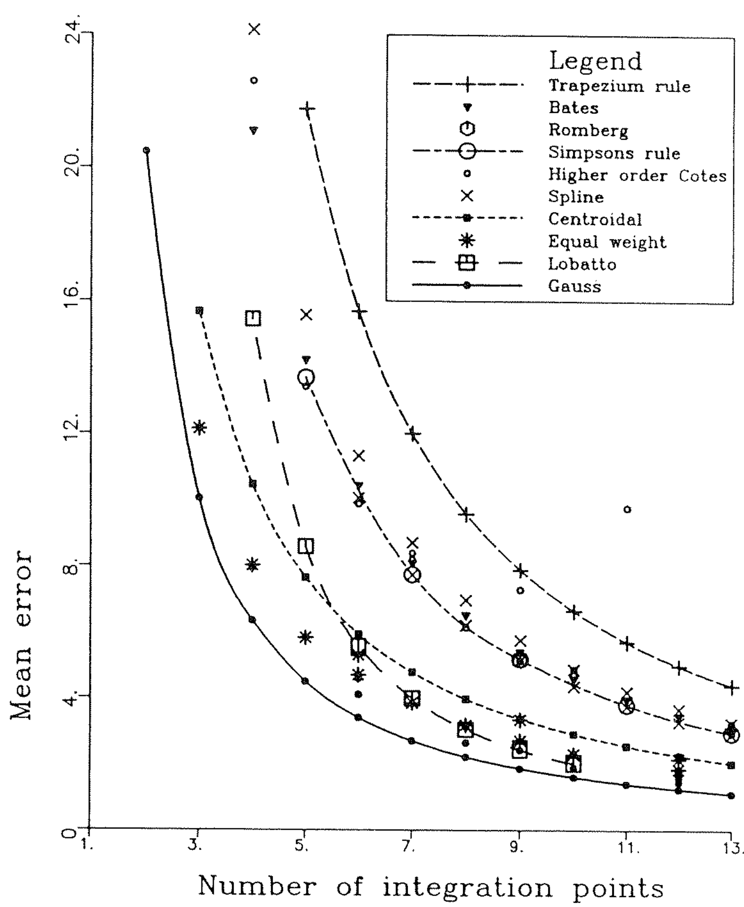

Figure 5. Comparison of numerical integration techniques

This result is important, since these methods allow integration over other ranges using the same interpolated formula. Although these problems are beyond the scope of this paper, it is reasonable to suppose that methods which give a good result for the present problem will be good methods for different ranges.

6. The 11-point Cotes method is less accurate than the 7-point Cotes method and the 10-point Cotes method is less accurate than the 4-point Cotes method applied 3 times, which also uses 10 integration points. The 9-point Cotes method is less accurate than the 8-point Cotes method, Simpson's rule using 9 points and spline curves using 9 points. These results accord with the observations above about the expected accuracy of the higher order Cotes formulae.
7. The Romberg method is slightly better overall than the 9-point version of Simpson's rule and the 9-point spline, but is marginally worse than the 5-point Cotes formula applied twice. These 4 methods, which all use 9 integration points, have very similar errors in most cases.
8. The spline curve with zero curvature at the ends is never better overall than the spline curve with the end curvature determined by extrapolation, but in individual cases it may give a better result.

9. Equal weight methods do not appear to offer any advantage in the present problem. The 3-point equal weight method repeated 3 times is more accurate than the 9-point equal weight method used once, which is probably due to the uneven distribution of the abscissae (Figure 2). It is also more accurate than the 9-point centroidal rule, despite the apparent similarity in the abscissae noted in Figure 2.
10. The modified trapezium rule, derived from the Bates method, shows a considerable improvement over the basic trapezium rule, but it is not as accurate as Simpson's rule (where applicable), or the spline curve, using the same number of points.
11. The analytical trapezium rule shows a marginal improvement over the basic trapezium rule, but this does not appear to be sufficient to repay the additional work involved in performing the calculations.

From the figure, it is also evident, for example, that 11 points are needed with Simpson's rule to achieve the same accuracy as the 5-point Gauss rule. It also shows that a 'law of diminishing returns' applies, at least to Gaussian integration, once 9 integration points are used.

CONCLUSIONS

A method has been devised for testing the overall performance of numerical procedures that relate to the integration of stresses through the thickness of plates and shells when there are discontinuities in the stresses. From the results of these tests the following can be concluded.

1. Use Gauss' method if integration is always required over the same range, and use as high an order formula as possible rather than making repeated use of simpler formulae.
2. Use Lobatto's method if it is essential to have an integration point at the ends of the integration range.
3. When the integration points are fixed at regular intervals, or for the more general problem, in which it is necessary to perform numerical integration over different ranges with the same integration points, techniques based on interpolation methods must be used.

   The spline method gives good results for all cases. Simpson's rule and the 5-point Cotes rule can both be used either once or repeatedly. The even order Cotes methods (trapezium, 4-point, 6-point and 8-point) are all worth using once, but there is no benefit in using them repeatedly. Romberg's method gives good results for 9 points.

The tests that have been used involve an equal weighting for all positions of the stress-discontinuity. In order to capture the important features of the responses of particular structures, it may be that some positions of the discontinuity are of more importance than others. It is intended to address this issue, and also that of the importance of having integration points on the surface, in future work.

REFERENCES

1. L. A. Louca, 'To propose a strategy for the numerical integration of stresses through the thickness of a plate or slab, in which there are discontinuities in the stress variation', Unpublished *M.Sc. Dissertation*, University of London, 1986.
2. H. Armen, A. B. Pifko, H. S. Levin and G. Isakson, 'Plasticity', in H. Tottenham and C. Brebbia (eds.), *Finite Element Techniques in Structural Mechanics*, Southampton University Press, 1970, Chapter 8, pp. 209–257.
3. E. Aldstedt and P. G. Bergan, 'Nonlinear time dependent concrete frame analysis', *J. Struct. Div. ASCE*, **104**, 1077–1092 (1978).
4. E. P. Popov, M. Khojasteh Bakht and S. Yaghmai, 'Bending of circular plates of hardening materials', *Int. J. Solids Struct.*, **3**, 975–987 (1967).
5. J. Backlund and Wennestrom, 'Finite element analysis of elastoplastic shells', *Int. j. numer. methods eng.*, **8**, 415–424 (1974).

6. E. Hinton, H. H. Abdel Rahman and H. Anderson, 'Computer-aided analysis of reinforced concrete slab and box-girder bridge decks on mainframes and micros', in F. Damjanic *et al.* (eds.), *Computer Aided Analysis and Design of Concrete Structures, Part 1*, Pineridge Press, Swansea, 1984, pp. 487–507.
7. E. Ramm and A. Matzenmiller, 'Computational aspects of elasto-plasticity in shell analysis', in D. R. J. Owen *et al.* (eds.), *Computational Plasticity, Models, Software and Applications, Part 1*, Pineridge Press, Swansea, 1987, pp. 711–735.
8. H. Parisch, 'Large displacements of shells including material nonlinearities', *Comp. Methods Appl. Mech. Eng.*, **27,** 183–214 (1981).
9. P. V. Marcal and W. R. Pilgrim, 'A stiffness method for elasto-plastic shells of revolution', *J. Strain Anal.*, **1,** 339–350 (1966).
10. J. A. Stricklin, W. E. Haisler and W. A. Von Riesman, 'Computation and solution procedures for non-linear analysis by combined finite element–finite difference methods', *Comp. Struct.*, **2,** 995–974 (1972).
11. M. A. Crisfield, 'Large-deflection elasto-plastic buckling analysis of plates using finite elements', *Transport and Road Research Laboratory Report LR 593*, 1973.
12. I. Cormeau, 'Elasto-plastic thick shell analysis by viscoplastic solid finite elements', *Int. j. numer. methods eng.*, **12,** 203–227 (1978).
13. B. Irons, *Techniques of Finite Elements*, Ellis Horwood, Chichester, 1980.
14. M. A. Crisfield, 'Finite element and mechanism methods for the analysis of masonry and brickwork arches', *Transport and Road Research Laboratory Report 19*, 1985.
15. P. G. Bergan, 'Some aspects of interpolation and integration in nonlinear finite element analysis of reinforced concrete structures', in F. Damjanic *et al.* (eds.), *Computer Aided Analysis and Design of Concrete Structures, Part 1*, Pineridge Press, Swansea, 1984, pp. 301–316.
16. C. E. Froberg, *Introduction to Numerical Analysis*, Addison-Wesley, Reading, Massachuesetts, 1969.
17. Z. Kopal, *Numerical Analysis*, Wiley, New York, 1955.
18. V. I. Krylov, *Approximate Calculation of Integrals*, Translated (from Russian) by A. H. Stroud, Macmillan, New York, 1959.
19. A. N. Lowan, N. Davids and A. Levenson, 'Table of zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula', *Bull. Am. Math. Soc.*, **48,** 739–743 (1942).
20. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series, Vol 55, Washington, 1955.
21. R. Radau, 'Etude sur les formules d'approximation qui servent a calculer la valeur numerique d'une integral definie', *J. Math. Pures Appl.*, **6,** 283–336 (1880).
22. S. N. Bernstein, 'On quadrature formulae of Cotes and Chebyshev' (in Russian), *Dokl. Akad. Nauk SSSR* **14,** 323–326 (1937).
23. R. W. Hamming, *Numerical Methods for Scientists and Engineers*, McGraw-Hill, New York, 1962.
24. W. Romberg, 'Vereinfachte numerische Integration', *Det. Kgl. Norske Vid. Selsk. Forh.*, **28,** 30–36 (1955).
25. Stiefel-Ruthishauser, 'Remarques concernant l'integration numerique', *Comptes Rendus (Paris)*, **252,** 1889–1900 (1961).
26. D. N. Bates, 'The mechanics of thin walled structures with special reference to finite rotations', *Ph.D. Thesis*, University of London, 1987.
27. K. W. Brodlie, Methods for drawing curves', in R. A. Earnshaw (ed.), *Fundamental Algorithms for Computer Graphics*, NATO ASI Series F, **17,** 1985, pp. 303–324.
28. A. K. Cline, 'Scalar- and planar-valued curve fitting using splines under tension', *Commun. Ass. Comp. Manu.*, **17,** 218–223 (1974).
29. J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *The Theory of Splines and Their Applications*, Academic Press, New York, 1967.