Cable design for continuous prestressed concrete bridges

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The Paper discusses the criteria which govern the cable design in continuous prestressed concrete beams. A rigorous analysis is used to show that Low's condition regarding the shortening of the extreme fibres is a particular case of a more general condition on the existence of a line of thrust within the allowable zone permitted by the stress conditions. An example of how these conditions can be applied in a continuous beam with three different spans is given.

Notation
Position measured positive downwards from the centroid
Tensile stresses are positive
Sagging moments are positive

\( A \) area
\( c_1 \) highest position of cable allowing for cover
\( c_2 \) lowest position of cable allowing for cover
\( d \) permissible range of position of cable (allowing for cover)
\( e \) eccentricity
\( e_m \) eccentricity of line of thrust at mid-span
\( e_{\text{max}} \) lower limit on eccentricity
\( e_{\text{min}} \) upper limit on eccentricity
\( e_p \) eccentricity of line of thrust
\( e_{pl} \) eccentricity of line of thrust at left pier
\( e_{pr} \) eccentricity of line of thrust at right pier
\( e_s \) eccentricity of cable profile
\( E \) Young's modulus
\( f_{ct} \) permissible stress in compression at transfer
\( f_{cw} \) permissible stress in compression at working load
\( f_t \) permissible stress in tension at transfer
\( f_w \) permissible stress in tension at working load
\( I \) second moment of area about centroid
\( l_h \) lever arm in hogging bending
\( l_s \) lever arm in sagging bending
\( L \) span
\( M_a \) minimum working load moment
\( (M_a)_{pl} \) \( M_a \) at left pier
\( (M_a)_{pr} \) \( M_a \) at right pier
\( M_b \) maximum working load moment
\( (M_b)_{m} \) \( M_b \) at mid-span
\( M_r \) moment range in one span (mid-span sagging - pier hogging)
\( M_{ls} \) mean live load moment in one span
\( M_r \) moment range at one section (sagging - hogging)

Written discussion closes 18 May 1988; for further details see p. ii.
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\[ M_1 \] moment at transfer
\[ P_{\text{max}} \] maximum possible prestress at one section
\[ P_{\text{min}} \] minimum possible prestress at one section
\[ P_1 \] prestressing force at transfer
\[ P_2 \] minimum prestress to satisfy moment range
\[ P_3 \] minimum prestress to satisfy lever arm
\[ P_4 \] minimum prestress for existence of line of thrust
\[ Q_j \] secondary moment at internal support \( j \)
\[ R \] (working load prestress)/(transfer prestress) \( = 1 - \text{losses} \)
\[ t_1 \] linear transformation at left pier
\[ t_m \] linear transformation at mid-span
\[ t_r \] linear transformation at right pier
\[ x \] position in span
\[ y \] position in beam
\[ y_1 \] position of top fibre (always - ve)
\[ y_2 \] position of bottom fibre (always + ve)
\[ Z_1 = 1/y_1 \] (always - ve)
\[ Z_2 = 1/y_2 \] (always + ve)
\[ \beta_j \] distribution coefficient for \( Q_j \)

Introduction
The design of continuous prestressed concrete bridges gives the engineer considerable freedom when compared with the design of statically determinate structures. By varying the secondary (or parasitic) moments, the relative magnitudes of the bending moments at mid-span and pier positions can be varied, thus allowing structures to be designed in which the line of thrust of the cable is outside the section, while the cable itself lies safely within the section.

2. The penalty to be paid for this freedom is greater complexity in design. It is necessary to determine the line of thrust of the actual cable that is used, or to choose a cable profile that is concordant. Either way, a knowledge and thorough understanding of the secondary moments in the structure are required.

3. Virtually all the standard texts on prestressed concrete design\(^1\)\(^-\)\(^3\) dwell on the problems associated with the calculation of the line of thrust and secondary moments associated with a given cable profile, but do not consider in detail the wider implications for the designer. However, one Author has considered other aspects of the problem. Low\(^4\)\(^,\) \(^5\) determined limits on the cable forces for the internal spans of a multi-span structure, based on considerations of potential crack patterns.

4. This Paper generalizes these ideas to include both the internal and end spans of a wider class of structures; those with unequal spans and those in which compressive stress limits govern.

Low’s work
5. Low considered a typical internal span of a multi-span viaduct and distinguished four governing conditions which he expressed as limits on the minimum prestressing force within that span. These conditions were derived on the assumption that the beams were limited by tensile stresses, which can be visualized by potential crack patterns if the limits are exceeded (Fig. 1). The four cases can be summarized as follows.
Fig. 1. Low's limits on cable forces (after reference 4): (a) limited by moment range \((P_1)\); (b) limited by range of eccentricity \((P_2)\); (c) limited by extension of extreme fibres \((P_3)\); (d) as (c), but limited by cover over pier \((P_4)\)
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Case 1: governed by moment range

6. The beam cracks at the bottom under the maximum sagging load and at the
top under the minimum sagging load, at the same section. The minimum prestress-
ing force is shown to be a function of the live load moment range \( M_l \), and is
independent of the parasitic moment.

\[
P_1 = \frac{M_l A}{(Z_1 - Z_2)}
\]  

(1)

Case 2: governed by lever arms

7. The beam cracks at the bottom at mid-span under the maximum moment
and at the top over the piers under minimum moment. The limiting value of the
minimum prestressing force is shown to be a function of the full moment range of
the span \( M_f \) and the maximum lever arms within the section.

\[
P_2 = \frac{M_f}{(l_s + l_b)}
\]  

(2)

It is implicit in this result that the parasitic moment has a specified value, in order
to make maximum use of the section depth.

Case 3: governed by shortening of the extreme fibre

8. The span is assumed to be a typical internal span, so there must be no net
rotation along the span caused by the prestressing force. The prestressing force is
assumed to be sufficient just to eliminate the tensile stresses at all points in the
bottom fibre owing to the average maximum live load moment \( M_{ls} \). This leads to
a limiting minimum prestressing force

\[
P_3 = \frac{M_{ls} A}{Z_2}
\]  

(3)

One corollary of this result is that the cable is placed near the top of its allowable
zone everywhere. Low demonstrates that this is often the governing condition for
design.

Case 4

9. Case 4 is a special case derived on the same principles as case 3, but limited
by the condition that the cable cannot be placed at the top of its allowable zone
over the piers as this would fail to satisfy cover requirements. A more complicated
limiting value for the prestressing force results, which need not concern us here.

10. Cases 1 and 2 are well established in practice and in texts, but case 3 was,
as far as the Author can establish, novel. It was the desire to put case 3 on a more
rigorous theoretical foundation which led to the work presented in this Paper. In
order to establish common ground between the various cases, and ‘conventional’
views of prestressing, it is necessary to review, briefly, the main principles of
section and cable design.

Magnel diagram and equations

11. The design of prestressed concrete beams at the working load is governed
by stress criteria (see Appendix 1). These can be rearranged into eight inequalities
of the general form

\[ e \geq -\frac{Z}{A} - \frac{Zf}{P} + \frac{M}{P} \]  \hspace{1cm} (4)

These relationships form straight bound lines on a plot of \( e \) against \((1/P)\), a construction known as the Magnel diagram. Each line passes through one of the Kern points \((e = -Z_1/A\) or \(-Z_2/A)\) when \(1/P = 0\) \((P = \infty)\). Therefore, four (and only four) of the inequalities define a feasible region which gives all valid combinations of force and eccentricity. Which four of the possible eight equations govern depends on the circumstances, but two will relate to the top fibre and two to the bottom fibre. Of each pair, one will be a tension limit and the other a compression limit.

12. The condition that the feasible region exists, which is an overriding condition in all cases (although not necessarily the governing one), is that the range of stresses in a particular fibre is less than the permissible stress range. These conditions can be expressed as limits on the elastic section moduli \((Z_1\) and \(Z_2)\). These inequalities are detailed in Appendix 1. It will be assumed henceforth that a section has been chosen already such that all these conditions are satisfied at all points along the length of the beam.

13. A typical Magnel diagram is shown in Fig. 2. The minimum and maximum

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**Fig. 2. Typical Magnel diagram**
prestressing forces $P_{\text{min}}$ and $P_{\text{max}}$ can be identified. If the chosen prestressing force is close to $P_{\text{max}}$, then compressive stresses will govern, while if $P$ is close to $P_{\text{min}}$, then tensile stress conditions will govern. Because minimizing the prestressing force usually results in greater economy, it will normally be found that a design is limited by tensile stresses.

14. However, if a section is chosen that only just satisfies the limiting conditions on the elastic modulus, a different result may be obtained. Suppose a road bridge is being considered, which will necessarily have a large top slab to carry the traffic. Local bending criteria will normally govern, so the section will have a top fibre modulus that is larger than required by the global flexural conditions. There is no reason why the bottom slab need be any larger than required to satisfy flexural criteria, so it is found that the Magnel diagram can look like that shown in Fig. 3.

15. The feasible region is long, but thin, and for most allowable prestressing forces is bounded by one tension limit and one compressive limit, even when $P$ is

Fig. 3. Magnel diagram for a road bridge

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close to \( P_{\text{min}} \). A similar result obtains for trough beam railway bridges, in which the rails are mounted on the bottom slab, which therefore is large, and it is the top flange which can be kept small.

16. The applied moments vary along the length of the beam. We can thus calculate \( P_{\text{min}} \) and \( P_{\text{max}} \) at all positions along the length of the beam, and plot these as functions of position. \( P_{\text{min}} \) will be largest at mid-span and over the piers, which are the positions where the moment range is largest, so the feasible region of the Magnel diagram is smallest. Similarly, \( P_{\text{max}} \) will be smallest at these positions. If the tensile stress limits are put to zero, then \( P_{\text{min}} \) is identical to Low’s value \( P_1 \).

**Line of thrust design**

17. The parasitic moments and their implication for design now need to be considered. Generally, any prestressing cable profile will induce flexural deflexions in the beam: if the beam is statically determinate, these cause no additional moments, as they are not resisted by the supports; if the beam is indeterminate, the tendency to deflect is resisted by the supports, whose reactions cause additional moments in the beam, known as secondary or parasitic moments \( M_2 \). However, because these moments are neither necessarily small (as implied by secondary) nor necessarily deleterious (as implied by parasitic), both terms are confusing.

18. If the actual position of the cable is given by \( e_s \), it can be stated that the cable appears to act at its line of thrust \( e_p \), where

\[
P e_p = P e_s - M_2
\]

so that

\[
e_p - e_s = -M_2/P
\]

19. The calculation of \( M_2 \), and hence \( e_p \), for a particular cable profile is straightforward. The forces that the cable exerts on the concrete can be calculated, and the response of the structure to these loads determined by a continuous beam program. This equivalent load method is well described in many texts (see, for example, reference 3).

20. Alternatively, and for our purposes, more conveniently, the secondary moments can be determined by considering the cable profile directly, using the principle of virtual work. The relevant equations are given in Appendix 2. The benefit for our purposes is that the method combines the structural analysis with the determination of the secondary moments, so we end up with a direct relationship between the cable profile and the secondary moments.

21. A few definitions will be required, but they are well known and are derived elsewhere. If \( M_2 = 0 \), \( e_p \) and \( e_s \) coincide, and the profile is concordant. All \( e_p \) are themselves concordant profiles. Since \( M_2 \) is the result of a set of self-equilibrating support reactions, it varies linearly between supports; the difference \( e_p - e_s \) will also vary linearly if the prestressing force is constant, must be zero at end supports, and is known as a linear transformation. Once \( e_p \) is known, any linear transformation will give a cable profile with the same line of thrust.

22. Two design philosophies can be identified, but they are equivalent.

(a) If the secondary moments are considered as prestressing effects, the stress conditions which lead to the Magnel diagram give limits on \( e_p \). For any chosen prestressing force, these can be plotted along the length of the beam; the designer has to find a concordant profile that satisfies these
limits. A suitable linear transformation can be determined to find a practical cable profile that satisfies cover requirements.

(b) If the designer has some estimate of the secondary moments, they can be regarded as loads, and included in the moments from which the Magnel diagram is calculated. The eccentricity limits are thus limits on $e$, and the problem becomes one of finding an actual cable profile which satisfies these conditions and causes the assumed secondary moments.

In practice, the choice between these methods is one of personal preference; the present Author finds the first easier, but knows many engineers who claim that they do not ‘bother about concordant profiles’, yet still correctly deal with secondary moments and indeed use them to considerable advantage.

23. For present purposes the first method will be used, as it requires no a priori knowledge of the secondary moments. The eccentricity inequalities (equations (25)–(32)) will be used to produce envelopes of the permissible line of thrust. The upper limit on the cable position will be termed $e_{\text{min}}$ and the lower limit $e_{\text{max}}$. These limits may be governed by different conditions at different positions along the beam, and will be functions of the chosen prestressing force.

**Design for limiting eccentricity**

24. For any particular structure, with specified spans and loading, limits on the line of thrust $e_p$ can be determined. A typical allowable zone will look like that shown in Fig. 4. The line of thrust will be high over the piers, and low at mid-span, as would be expected.

25. The effect of applying a linear transformation to the limiting values of $e_p$ can now be considered. If a sagging secondary moment is chosen, the effect will be to produce a cable profile $e_s$ which is lower than $e_p$. This can be an advantage: cover requirements for $e_s$ have to be satisfied, but not for $e_p$. Therefore, it does not matter if the line of thrust lies outside the section over the piers, provided that the corresponding cable is within the section.

26. What range of secondary moments is permissible? In the case of the $e_p$
limits shown in Fig. 4, the secondary moments must be large enough to bring the
cable into the section (and allow for cover) over the piers, but not so large that the
cable loses cover at the bottom at mid-span. Bearing in mind that the form of the
secondary moment variation is known along the beam, in that it is zero at pinned
ends and varies linearly between supports, there is either an infinity of possible
solutions, a unique solution or no solution. The condition that there is a unique
solution needs to be determined, as this represents the dividing line between a
feasible and an unfeasible design.

27. Consider the single internal span shown in Fig. 5(a). The existence of
symmetry will not be assumed for the moment. A positive linear transformation is
required at both pier positions, to bring the cable within the allowed zone, but a
larger value is required at the left-hand pier. However, the linear transformation
must not be too large near mid-span. The two limits are shown in Fig. 5(b) as
functions of position within the span. The difference \( e_p - e_s \) varies linearly along
the span, and, as drawn, many possible lines can be fitted between the bounds. The
condition that one line can just be fitted between the bounds on the transform-
ation is being sought. This will occur when a point on the upper bound is just
tangential to the line joining the two values over the piers, as shown in Fig. 5(c).

28. Let the position within the span where the transformation is tangential be
at a distance of \( x \) from the left-hand end. Then

\[
t_m \geq \frac{x}{L} t_r + \frac{(L-x)}{L} t_1
\]

(7)

and

\[
t_r = c_1 - e_{pr}
\]

\[
t_1 = c_1 - e_{pl}
\]

\[
t_m = c_2 - e_m
\]

(8)

which can be rearranged to give

\[
d \geq e_m - e_{pr} \frac{x}{L} - e_{pl} \frac{(L-x)}{L}
\]

(9)

29. In any particular design, the stress conditions which govern at each section
will be known; as there are four possibilities for each of \( e_m \), \( e_{pr} \) and \( e_{pl} \), there are 64
different combinations, so it is not sensible to detail all of them here. However, we
can consider one case as an illustrative example.

30. If it is assumed that the working load conditions govern at each point, this
gives

\[
e_{pl} = - \frac{Z_1}{A} - \frac{Z_1 f_{tw}}{RP_t} + \frac{(M_{a})_{pl}}{RP_t}
\]

\[
e_{pr} = - \frac{Z_1}{A} - \frac{Z_1 f_{tw}}{RP_t} + \frac{(M_{a})_{pr}}{RP_t}
\]

\[
e_m = - \frac{Z_2}{A} - \frac{Z_2 f_{tw}}{RP_t} + \frac{(M_{a})_{m}}{RP_t}
\]

(10)
Fig. 5. Linear transformation for internal span: (a) typical internal span (L); (b) permissible linear transformations—general case; (c) permissible linear transformation—limiting case
Substituting into equation (9) and rearranging, gives

$$RP_t \geq \frac{\left[ \left( \frac{M_{b,m}}{L} - \frac{x}{L} (M_{a,pr}) - \frac{(L-x)}{L} (M_{a,pl}) \right) - f_{tw}(Z_2 - Z_1) \right]}{\left( d + \frac{(Z_2 - Z_1)}{A} \right)} \tag{11}$$

If the effects of losses are ignored, $f_{tw} = 0$, and the beam is symmetrical, so that $(M_{a,pr} = (M_{a,pl}$ and $x = 0.5$, equation (11) reduces to Low's much simpler expression for $P_2$.

31. In an end span (say at the left-hand end of the beam), $t_i$ must be zero.

$$t_m \geq \frac{x}{L} \mu t$$ \tag{12}

so that

$$e_m \geq \left( c_2 - \frac{x}{L} c_1 \right) + \frac{x}{L} e_{pr} \tag{13}$$

which leads to

$$RP_t \geq \frac{\left[ \left( \frac{M_{b,m}}{L} - \frac{x}{L} (M_{a,pr}) - f_{tw}(Z_2 - \frac{x}{L} Z_1) \right) \right]}{\left( c_2 - \frac{x}{L} c_1 \right) + \frac{(Z_2 - (x/L)Z_1)}{A}} \tag{14}$$

A similar result can be derived for right-hand end spans.

32. In general in internal spans, and in all cases in end spans, the value of $x$ is unknown. However, it is easy to determine which stress condition governs the position of the line of thrust at the piers, and also at intermediate positions, and to produce similar equations in all cases.

33. It should be appreciated that these results are a relaxation of the conditions that apply for statically determinate beams, in which secondary moments cannot occur; the limits on cover must be applied at every section to the line of thrust of the cable, which now coincides with the actual position.

**Conditions on existence of line of thrust**

34. Consideration has not yet been given to the question of whether it is actually possible to find a line of thrust $e_p$ which satisfies the stress limits. In the general case, the line of thrust will be constrained to lie between bounds $e_{\text{min}}$ and $e_{\text{max}}$ defined by the various forms of equation (25). Typically, the bounds will rise over piers but will lie below the centroid in regions where the bending moments are sagging. Provided that the prestressing force has been chosen so that it lies within the feasible region of the Magnel diagram for all positions along the beam, it follows that $e_{\text{max}} \geq e_{\text{min}}$ for all positions.

35. A cable placed at the lowest position $e_{\text{max}}$ will cause the maximum possible sagging secondary moments $M_{2,\text{max}}$. (Imagine that the beam is made statically determinate by the removal of the internal supports. The effect of the cable's acting alone will be to induce the maximum hogging curvature in the beam and hence the
largest uplift at the intermediate support positions. It will then require the largest
downward forces at the intermediate positions to restore the beam to its unde-
flected position at the supports, thus maximizing $M_2$.

36. For normal beams, subjected to gravity loads applied within the length of
the beam, the resulting bending moment diagram will normally have larger
sagging regions than hogging regions, so the corresponding secondary moments
are normally sagging.

37. By a similar argument, a cable placed at $e_{\text{min}}$ will cause the maximum
hogging (or minimum sagging) secondary moment $(M_{2}\text{min})$. If the maximum and
minimum secondary moments are both sagging, it follows that it will be impos-
sible to find an intermediate cable profile that causes zero secondary moments; no
line of thrust can thus exist.

38. This is the essence of Low’s third condition on the prestressing force $P_3$. If
the prestressing force is too low, no concordant line of thrust will be found to exist;
a search for it would be fruitless. Low’s condition is based on the assumption that
there must be sufficient prestress to ensure that the bottom fibre does not
decompress, and that the total shortening of the bottom fibre due to the prestress
is the same as the shortening of the centroid. There will thus be no net rotation
along the span which is necessary to allow multiple spans all to have the same
structural form and cable layout. More simply, this is a compatibility condition to
ensure that individual spans in the viaduct fit together.

39. The condition for concordancy of the line of thrust is a similar compat-
ibility condition, although usually expressed in terms of displacement rather than
slope, at the supports. Despite that difference, they are the same condition in
practice. The present condition is more general. It applies to the whole structure,
including end spans, and can take account of the fact that different stress condi-
tions apply at different positions along the beam.

40. There are expressions for $e_{\text{min}}$ and $e_{\text{max}}$ at all positions along the length of
the beam, from equations (25)–(32) and for any chosen prestressing force, the
secondary moments at each pier position can be determined, using equations (47).
It is thus straightforward in the design process to try various prestressing forces
until the maximum and minimum secondary moments at each pier position are of
opposite signs, indicating that a concordant line of thrust exists and can be sought.

41. However, it would be beneficial to be able to find conditions which must be
satisfied to enable us to find the minimum prestressing force directly. $M_{2}\text{max}$ will
normally be sagging in all cases, but $M_{2}\text{min}$ can be sagging or hogging. Therefore,
the dividing line between these two cases, which will occur when $e_{\text{min}}$ itself defines
a concordant profile, has to be found. Can any condition which must be satisfied
for $e_p$ to exist be determined?

42. Consider equation (47) which defined the secondary moments $Q_i$ at each
internal support. If the cable profile is concordant, then all $Q_i$ are zero. It is both a
necessary and a sufficient condition for this result that the right-hand sides are all,
separately, zero. Thus it can be shown that a cable profile is concordant if it can be
shown that

$$
\int \frac{\beta_x P e}{EI} \, dx = 0 \quad \text{for all } i
$$

(15)

This result is perfectly general, although not particularly useful in its present form.

43. If it is assumed that both $EI$ and $P$ are constant everywhere, then it is only
necessary to show that
\[ \int \beta_i e \, dx = 0 \quad \text{for all } i \]  \hspace{1cm} (16)

Although these integrals are formally for the complete structure, in practice, for any \( i \), the \( \beta_i \) values are only non-zero in the spans adjacent to the \( i \)th support. A sufficient (but not necessary) condition to ensure concordancy of \( e \) is that \( \int \beta_i e \, dx \) is zero for each span separately.

44. Consider a beam whose internal spans are of roughly equal size and whose end spans are so proportioned that the maximum hogging moments over all supports are approximately the same. This is a common layout for multi-span structures, although there are many exceptions.

45. For internal spans it is common practice to make the cable profiles identical within each span, and symmetric about the mid-span. \( \beta_i \) is skew-symmetric about the mid span, where it takes the value 0.5. Thus \( \int \beta_i e \, dx = 0.5 \int e \, dx \), so we only need to show that the average value of the eccentricity is zero within the span.

46. It is usual, although again not invariable, that the value of \( e_{\text{min}} \) will be governed by the tensile stress limit in the bottom fibre under the action of the maximum moment \( M_b \). Therefore, \( e \) will satisfy identically the condition that
\[ e = -\frac{Z_2}{A} \frac{Z_2 f_{tw}}{RP_t} + \frac{M_b}{RP_t} \]  \hspace{1cm} (17)

Thus, if \( \int e \, dx = 0 \)
\[ L \left( -\frac{Z_2}{A} \frac{Z_2 f_{tw}}{RP_t} \right) + \frac{1}{RP_t} \int M_b \, dx = 0 \]  \hspace{1cm} (18)

where \( L \) is the span.

47. If the average value of \( M_b \) over the span is \( \bar{M}_b \), then the condition can be rearranged to give
\[ RP_t = \frac{\bar{M}_b - Z_2 f_{tw}}{Z_2/A} \]  \hspace{1cm} (19)

It is relatively easy to show that this is a minimum condition on the prestressing force. If \( P_t \) is less than this value, \( e_{\text{min}} \) will give rise to a sagging secondary moment, and no concordant profile can exist.

48. This expression is a general form of Low’s limit on \( P_t \). Low set \( f_{tw} \) to zero, and his expression uses the mean value of the maximum live load moment, whereas the Author’s expression uses the mean value of the maximum total moment. However, he postulates a beam in which, owing to dead load, no rotation occurs over the piers; in that case, the mean value of the dead load moment will also be zero.

49. For end spans, slightly different conditions apply, as the results cannot be simplified by symmetry. Consider the first span, where we only need to show that \( \int \beta_1 e \, dx = 0 \). If we measure distance \( x \) from the end of the beam, up to the first support at \( x = L \), then \( \beta_1 = x/L \). If \( e_{\text{min}} \) is governed by the same stress condition as in the internal spans, then it needs to be shown that
\[ \int_0^L \frac{x}{L} \left( -\frac{Z_2}{A} \frac{Z_2 f_{tw}}{RP_t} + \frac{M_b}{RP_t} \right) \, dx = 0 \]  \hspace{1cm} (20)
or

\[-L\left(\frac{Z_2}{A} + \frac{Z_2 f_{tw}}{RP_t}\right) + \frac{1}{RP_t} \int_0^L x M_b \, dx = 0\]  \hspace{1cm} (21)

Integrating by parts, gives

\[L\left(\frac{Z_2}{A} + \frac{Z_2 f_{tw}}{RP_t}\right) = \bar{M}_b L - \frac{1}{L} \int_0^L \left( \int_0^x M_b \, ds \right) \, dx\]  \hspace{1cm} (22)

Defining

\[\bar{M}_b = \frac{1}{L^2} \int_0^L \left( \int_0^x M_b \, ds \right) \, dx\]  \hspace{1cm} (23)

this equation can be rearranged to give

\[RP_t = \frac{\bar{M}_b - \bar{M}_b - Z_2 f_{tw}}{Z_2/A}\]  \hspace{1cm} (24)

50. These conditions are generalized forms of those given by Low, and although they only apply precisely in certain circumstances, they are a useful guide to the minimum prestress required. The average values of \(M_b\) within each span and the double integral of \(M_b\) in the end spans are easy to carry out numerically, and it is therefore straightforward to include these conditions in the design process.

**Condition \(P_4\)**

51. The condition that Low calls \(P_4\) is a combination of the two cases \(P_2\) and \(P_3\). The limit which governs \(P_3\) is that \(e_{min}\) is a concordant profile, and so may be a line of thrust. Such a profile will have a larger range than that given if the line of thrust passes through \(e_{max}\) over the piers, so it may not be possible to fit a cable into the section.

52. Therefore, it is necessary to find the condition that a cable with the lowest permissible range of eccentricities will be a concordant profile. Low got round this by assuming that the actual cable profile was at its maximum practical negative eccentricity (i.e. \(c_1\)) over the piers, and remained at that eccentricity until forced down by the conditions on \(e_{min}\). This is slightly artificial in that the cable will have to be curved in practice, but the principle is valid, and will not be dealt with in detail here.

**Visualization of conditions**

53. The four conditions that have been discussed can be summarized by visualizing the allowable limits on \(e_p\).

- \(P_1\) The conditions on \(P_1\) ensure that a feasible zone for \(e_p\) exists, so that \(e_{max} > e_{min}\) (Fig. 6(a)).
- \(P_2\) The condition \(P_2\) ensures that the range of eccentricities within a span are less than allowable by considerations of cover (Fig. 6(b)).
- \(P_3\) The condition \(P_3\) ensures that a concordant profile exists between the limits \(e_{min}\) and \(e_{max}\) (Fig. 6(c)).
Fig. 6. Visualization of limiting force conditions: (a) $P_1$; (b) $P_2$; (c) $P_3$; (d) $P_4$
The condition $P_4$ ensures that a cable which satisfies the eccentricity range condition (as for $P_2$) will also satisfy the condition that a concordant line of thrust can exist (as for $P_3$) (Fig. 6(d)).

Example

54. As an illustration of some of the principles discussed previously, the design of a 3-span bridge, with spans of 40 m, 50 m and 30 m, will be considered. The structure is to carry three traffic lanes, each 3.65 m wide within an overall width of 13 m.

55. The loading is to be standard highway loading HA plus 45 units of HB, to the Department of Transport draft standard. The concrete will be assumed to have permissible stresses in compression of $-15$ N/mm$^2$ at transfer and of $-16.5$ N/mm$^2$ at the working load (tensile stresses are considered positive), with no tension being allowed in the preliminary design. The section will be prismatic. The live load envelopes are determined using an enveloping program based on Macaulay’s method (Fig. 7).

56. The largest moment ranges occur at the centre of left-hand side span (chainage 20 m—moment range 37040 kNm), and over the left-hand pier (chainage 40 m—moment range 28193 kNm). These will thus be regarded as the critical positions for the section design.

57. The section design is based on the moment range at the critical sections, which leads to minimum section moduli of $-2.24$ m$^3$ for $Z_1$ and $2.64$ m$^3$ for $Z_2$. The chosen section is shown in Fig. 8; the overall depth is chosen from span : depth ratio considerations, while the top flange is fixed by the road width and is considerably larger than that required for purely flexural considerations. The bottom flange is only marginally above the minimum size.

58. Figure 9 shows the Magnel diagrams at the two critical positions (20 m

![Diagram](image)

Fig. 7. Moment envelopes used in design example

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Fig. 8. Cross-section chosen for design example

and 40 m). The minimum prestressing forces at the two sections are 40,449 kN and 33,538 kN respectively. The feasible region lies outside the section at the top at chainage 40 m, but the feasible region lies well within the section at 20 m, so it is probable that the cable can be brought within the section when linear transformations and the associated secondary moments are considered.

59. The requisite prestress force can now be considered. The first criterion is that a feasible region must exist on the Magnei diagram at each cross-section ($P_1$).

Fig. 9. Magnei diagrams at 20 m and 40 m from left-hand end
Fig. 10. Limits on prestressing force

The minimum and maximum prestressing forces (calculated by the standard elastic design inequalities (25)-(44) given in Appendix 1) are shown in Fig. 10. The maximum force is unlikely to be a governing criterion, but the minimum prestressing force must be at least 40.449 kN. However, the other two criteria must also be checked.

60. The values for $P_2$ calculated on the basis of the eccentricity range are

Fig. 11. Limits on line of thrust of cable $e_p$
shown in Table 1, as are those calculated for $P_3$ on the assumption that a concordant profile exists within the line of thrust zone. Clearly, a prestressing force of at least 51.822 kN is required. If a prestressing force of 52.000 kN is chosen, inequalities (25)-(32) can be used to determine limits on the line of thrust $e_p$ which are shown in Fig. 11.

61. As a check on the existence of a concordant profile, the secondary moments, that result if a cable at the highest position $e_{\text{min}}$ and the lowest position $e_{\text{max}}$ is chosen, can be calculated (see Table 2). Clearly, it is possible to find a concordant cable profile which fits within these limits, but we would expect it to lie close to the upper limit on $e_p$, especially in the left and centre spans.

62. A line of thrust can be sought which satisfies the limits on $e_p$. Use will be made of the principle that any bending moment diagram which corresponds to a real load on the structure will be a scaled line of thrust. A notional cable force of 1000 kN will be assumed, so that a bending moment of 1000 kNm corresponds to an eccentricity of 1 m; and a continuous beam program will be used to calculate the results.

63. The process is iterative, and takes several steps; a load is assumed, the bending moments calculated, and they are compared with the eccentricity limits. If the calculated eccentricity lies outside the limits, a modification to the applied loads is made, and the process repeated. The loads are also chosen to minimize the reaction at the internal supports, to eliminate kinks in the cables at these points. An automated version of this process will be presented separately.10

64. The first estimate was to take a uniformly distributed load over the whole structure, of intensity 7 kN/m. After a number of iterations, the loading was modified to have 8 components (see Table 3). The corresponding line of thrust is shown in Fig. 11. As expected, the cable profile is close to the upper limit over the main span, and for a fair proportion of the left-hand span.

65. The profile lies above the beam at the left-hand pier, but is well above the soffit at mid-span. A linear transformation can now be used to choose a cable profile that fits within the section. The cable profile needs to be lower than the line of thrust by 0.334 m at the left pier, and 0.079 m at the right pier, to give 0.15 m from the top surface to the centre of the cable. With a cable force of 52.000 kN, this

Table 1. Values for $P_2$ and $P_3$

<table>
<thead>
<tr>
<th>Span</th>
<th>$P_2$ : kN</th>
<th>$P_3$ : kN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>41 172</td>
<td>51 438</td>
</tr>
<tr>
<td>Centre</td>
<td>46 137</td>
<td>51 822</td>
</tr>
<tr>
<td>Right</td>
<td>28 520</td>
<td>27 731</td>
</tr>
</tbody>
</table>

Table 2. Secondary moments at internal supports

<table>
<thead>
<tr>
<th>$e_{\text{min}}$ (hogging)</th>
<th>$Q_1$ : kNm Left pier</th>
<th>$Q_2$ : kNm Right pier</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{\text{max}}$ (sagging)</td>
<td>16 029</td>
<td>15 941</td>
</tr>
</tbody>
</table>
Table 3. Notational loading to give \( e_p \)

<table>
<thead>
<tr>
<th>Load</th>
<th>Start</th>
<th>End</th>
<th>Intensity: kNm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>30.0</td>
<td>6.8</td>
</tr>
<tr>
<td>2</td>
<td>30.0</td>
<td>35.0</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>35.0</td>
<td>45.0</td>
<td>-25.6</td>
</tr>
<tr>
<td>4</td>
<td>45.0</td>
<td>55.0</td>
<td>5.2</td>
</tr>
<tr>
<td>5</td>
<td>55.0</td>
<td>82.5</td>
<td>7.0</td>
</tr>
<tr>
<td>6</td>
<td>82.5</td>
<td>95.0</td>
<td>-18.8</td>
</tr>
<tr>
<td>7</td>
<td>95.0</td>
<td>105.0</td>
<td>10.0</td>
</tr>
<tr>
<td>8</td>
<td>105.0</td>
<td>120.0</td>
<td>5.0</td>
</tr>
</tbody>
</table>

corresponds to secondary moments of 17,368 kNm and 4108 kNm (before losses), at these positions.

66. To demonstrate that the limits on the actual cable zone \( e_s \) are also satisfied, these secondary moments can be treated as loads, and the limits determined. These are shown in Fig. 12, together with the actual cable profile, and it can be seen that the conditions are satisfied. (It must be stated that this check is academic—if the limits on \( e_p \) are satisfied, then those on \( e_s \) will automatically be satisfied.)

Conclusion

67. It has been shown that a constraint on the existence of a feasible line of thrust, within the zone permitted by the stress criteria, must be considered when choosing the prestressing force in a statically indeterminate beam.

![Graph showing limits on actual cable profile \( e_s \)](image)

*Fig. 12. Limits on actual cable profile \( e_s \)"
68. This condition has been shown to be a generalization of Low's condition on the extension of the extreme fibre.

69. An example has been given demonstrating the use of this condition in practice, and illustrating the steps needed to design cable profile rationally and quickly.

Appendix 1. Stress limit criteria

70. At any cross-section, the beam will be loaded by a minimum working load moment $M_a$ and maximum moment $M_b$, which are applied after all prestressing losses. Before time dependent losses have occurred, it will be assumed that there is a fixed transfer moment $M_t$. If the stresses are limited to $f_t$ in tension and $f_c$ in compression at transfer, and $f_{cw}$ in compression at the working load, 12 distinct stress criteria can be identified, based on limiting tension and compression in both extreme fibres under the three load conditions. However, if it is insisted that $M_b > M_a$, this number can be reduced to eight, as four will automatically be satisfied.

71. These conditions can be rearranged to give inequalities on the permissible eccentricities, as functions of the applied moments, the section properties and the reciprocal of the prestressing force.

$$e \geq -\frac{Z_1}{A} - \frac{Z_1 f_t}{P_t} + \frac{M_t}{P_t}$$  \hspace{1cm} (25)

$$e \geq -\frac{Z_1}{A} - \frac{Z_1 f_{cw}}{RP_t} + \frac{M_b}{RP_t}$$  \hspace{1cm} (26)

$$e \geq -\frac{Z_2}{A} - \frac{Z_2 f_t}{P_t} + \frac{M_b}{P_t}$$  \hspace{1cm} (27)

$$e \geq -\frac{Z_2}{A} - \frac{Z_2 f_{cw}}{RP_t} + \frac{M_b}{RP_t}$$  \hspace{1cm} (28)

$$e \leq -\frac{Z_1}{A} - \frac{Z_1 f_t}{P_t} + \frac{M_t}{P_t}$$  \hspace{1cm} (29)

$$e \leq -\frac{Z_1}{A} - \frac{Z_1 f_{cw}}{RP_t} + \frac{M_a}{RP_t}$$  \hspace{1cm} (30)

$$e \leq -\frac{Z_2}{A} - \frac{Z_2 f_t}{P_t} + \frac{M_a}{P_t}$$  \hspace{1cm} (31)

$$e \leq -\frac{Z_2}{A} - \frac{Z_2 f_{cw}}{RP_t} + \frac{M_a}{RP_t}$$  \hspace{1cm} (32)

72. By considering the change in stress in a particular fibre, and comparing it with the permissible stress range, three limiting conditions for the elastic section modulus of each extreme fibre can be derived.

$$Z_1 \leq \frac{(M_a - RM_t)}{(R f_t - f_{tw})}$$  \hspace{1cm} (33)

$$Z_1 \leq \frac{(M_b - RM_t)}{(f_{cw} - R f_t)}$$  \hspace{1cm} (34)

$$Z_1 \leq \frac{(M_b - M_a)}{(f_{cw} - f_{tw})}$$  \hspace{1cm} (35)
\[ Z_2 \geq \frac{(M_b - RM_i)}{(Rf_{ct} - f_{iw})} \] \hspace{1cm} (36)

\[ Z_2 \geq \frac{(M_b - RM_i)}{(f_{cw} - Rf_{ct})} \] \hspace{1cm} (37)

\[ Z_2 \geq \frac{(M_a - M_b)}{(f_{cw} - f_{iw})} \] \hspace{1cm} (38)

73. Once a section has been chosen, a Magnel diagram can be drawn at the cross-section, and the maximum and minimum prestressing forces can be determined. Alternatively, by considering all combinations of top and bottom fibre stress limits, the following limits on the prestressing force can be derived:

\[ P_t \geq -\frac{Af_{tw}}{R} + \frac{A}{R(Z_2 - Z_1)} (M_b - M_a) \] \hspace{1cm} (39)

\[ P_t \geq \frac{A}{R(Z_2 - Z_1)} [(Rf_{ct} Z_1 - f_{iw} Z_2) + (M_b - RM_i)] \] \hspace{1cm} (40)

\[ P_t \geq \frac{A}{R(Z_2 - Z_1)} [(f_{cw} Z_1 - Rf_{ct} Z_2) + (RM_i - M_a)] \] \hspace{1cm} (41)

and

\[ P_t \leq -\frac{Af_{cw}}{R} - \frac{A}{R(Z_2 - Z_1)} (M_b - M_a) \] \hspace{1cm} (42)

\[ P_t \leq \frac{A}{R(Z_2 - Z_1)} [(Rf_{ct} Z_1 - f_{cw} Z_2) - (RM_i - M_a)] \] \hspace{1cm} (43)

\[ P_t \leq \frac{A}{R(Z_2 - Z_1)} [(f_{cw} Z_1 - Rf_{ct} Z_2) - (M_b - RM_i)] \] \hspace{1cm} (44)

These inequalities have been used in the Paper to provide upper and lower limits on the prestressing force \( P_{\text{min}} \) and \( P_{\text{max}} \).

**Appendix 2. Calculation of secondary moments by virtual work**

74. It is possible to determine the secondary moments on a beam directly from the cable profile, using the principle of virtual work. It is still necessary to solve a set of simultaneous equations (one for each internal support), but it is not necessary to convert the cable profile into the equivalent forces and then perform a structural analysis.

75. An equilibrium system consisting of a fictitious moment system and associated reactions, as shown in Fig. 13(a), is identified. The reactions \( R_{ik} \) are unknown but will not need to be determined. There will be one such equilibrium system for every internal support.

76. The compatibility system will be chosen to be the actual displacement of the beam. There will be a curvature (+ ve sagging) due to the horizontal component of the prestressing force \( P \) at an eccentricity \( e \) of \( -Pe/EI \). The - ve sign arises because a cable with + ve eccentricity causes a hogging curvature.

77. At each internal support there will be an actual secondary moment \( Q_i \), as shown in Fig. 13(b). Each of these secondary moments will give rise to a curvature of \( \beta_i Q_i/EI \), which varies along the length of the beam.

78. The virtual work equation will thus be

\[ \int_0^L M\kappa \, dx = \sum W\Delta \] \hspace{1cm} (45)
Fig. 13. Virtual work systems: (a) equilibrium system; (b) compatibility system

79. The only point forces on the beam in the equilibrium system are the reactions \( R'_{ik} \), but as these are applied at the supports, where \( \Delta \) is zero, then the right-hand side of the virtual work equation must be zero and we do not need to know the values of \( R'_{ik} \).

80. There will thus be one virtual work equation for each of the internal supports.

\[
\int \beta_i Q_i \left( \sum_j \beta_j Q_j - Pe \right) EI \, dx = 0 \quad i = 1, 2, \ldots, n
\]  

(46)

81. For any particular equation, \( Q_i \) is a constant, so it can be cancelled from the equation without loss of generality, which can be rearranged to give

\[
\sum_j Q_j \int \frac{\beta_i \beta_j}{EI} \, dx = \int \frac{\beta_i Pe}{EI} \, dx \quad i = 1, 2, \ldots, n
\]  

(47)

The equations form a set of linear equations in the unknown secondary moments \( Q_j \); the cable profile appears only on the right-hand side of the equations, and the coefficients of the \( Q_j \) terms are integrals of products of the \( \beta \) functions, many of which are zero. Of the non-zero terms, although they are formally integrals over the whole length of the beam, only the spans (at most two) over which both \( \beta_i \) and \( \beta_j \) are non-zero need to be considered.

82. Equations (47) are perfectly general, allowing the prestress force, eccentricity and stiffness to vary, but if \( EI \) is constant, the integrals on the left-hand side can be performed analytically. Simple algebra then yields

\[
\frac{1}{6} \begin{bmatrix}
2(L_1 + L_2) & L_2 & \cdots & \cdots & \cdots \\
L_2 & 2(L_2 + L_3) & L_3 & \cdots & \cdots \\
\cdots & L_3 & 2(L_3 + L_4) & L_4 & \cdots \\
\cdots & \cdots & L_4 & \text{etc.} & \\
\end{bmatrix} Q_j = \int \beta_i Pe \, dx = J_i
\]  

(48)

The right-hand side terms \( J_i \) can be found by numerical integration. Because the \( \beta_i \) terms are zero, other than in the spans adjacent to the support in question, the integrals for each \( J_i \) need only consider two spans.
References
Cable design for continuous prestressed concrete bridges

C. J. Burgoyne

Mr A. McC. Low, Ove Arup and Partners, London
This extension to the published theory on this topic is to be welcomed, and I have one comment. Fig. 10 shows the limits on prestressing force along the 3-span bridge example. The maximum force shown is calculated from the extremity of the Magnel diagram. This could be called $P_{1_{\text{max}}}$, What is missing is the maximum force limits imposed by case 3, $P_{3_{\text{max}}}$, Following arguments similar to those in § 8 of the Paper and in §§ 15–20 of reference 4, it can be shown that for an internal span

$$P_{3_{\text{max}}} = f_{cw}A - \frac{\bar{M}_{1\text{H}}A}{Z_{B}} = \frac{Z_{2}f_{cw} - \bar{M}_{3}}{Z_{2}/A}$$  \hspace{1cm} (49)$$

84. From Fig. 7, it may be judged that for the centre span $\bar{M}_{1\text{H}} = 0.5 \times \bar{M}_{1\text{S}}$, Hence $P_{3_{\text{max}}} = 6.616 \times 16.5 - 0.5 \times 51.82 = 83.3$ MN. This is much less than the values of $P_{\text{max}}$ plotted in Fig. 10. A similar derivation is possible for the end spans.

85. Introducing the concept of maximum cable force in continuous spans raises some interesting questions. Are cases 1 and 3 the only cases to impose maximum limits? Cases 2 and 4 are both governed by the limit on the cable profile amplitude. When there is an excess of force this will not govern. Hence cases 2 and 4 govern $P_{\text{min}}$ but not $P_{\text{max}}$. Are there complementary cases 5 and 6 which govern $P_{\text{max}}$ but not $P_{\text{min}}$?

Dr Burgoyne
It is quite correct to state that other criteria could be derived. In the Paper, equation (19) was derived on the assumption that $e_{\text{min}}$ is the limiting position of the line of thrust, and that this condition is governed by tensile stresses in the bottom fibre under the action of the maximum applied moment $M_{b}$. Mr Low’s condition (49) is derived on the assumption that $e_{\text{max}}$ is the limiting position, and that this is governed by compressive stresses in the bottom fibre under the action of the minimum applied moment $M_{a}$.

87. Clearly, there are many other similar relationships that could be derived, but this should not be allowed to detract from the main point of the argument: the prestressing force must be chosen such that a cable placed at $e_{\text{min}}$ would give rise to hogging secondary moments, while one placed at $e_{\text{max}}$ would give rise to sagging secondary moments. Under those circumstances, a concordant line of thrust between $e_{\text{max}}$ and $e_{\text{min}}$ can exist.
DISCUSSION

88. Mr Low raises the possibility that upper limits might exist on the cable force that correspond to the lower limits imposed by $P_2$ and $P_4$. This is not the case, however, since $P_2$ and $P_4$ are limited by the maximum eccentricity determined by the section dimensions. Increasing the cable force decreases the eccentricity, and there are no corresponding lower limits on eccentricity that need to be introduced. There is no need to worry about possible limits on $P_5$ and $P_6$. 