Prestressed beams lifting off elastic supports

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Synopsis

The paper presents a method of determining the variation of bending moment within a prestressed concrete beam as it lifts off falsework; the falsework reactions can also be calculated.

The falsework is assumed to provide an elastic restraint if the beam deflects downwards, but no restraint if the beam lifts off.

The problem is non-linear, since it is not known ab initio *which parts of the beam go up and which down. The paper presents a method of analysing a beam for these support conditions, under a generalised loading.*

Results are presented for a three-span beam as it progressively lifts off its falsework as more cables are stressed; the effects of varying the falsework stiffness are considered.

The case of a beam being prestressed in such a way that excessive falsework loads can be caused is also presented.

Notation

(Some minor notation is used once only and is defined when used in the text)

y is the displacement
θ is the slope
M is the bending moment
Q is the shear force
x is the distance along beam
EI is the flexural stiffness of beam
k is the support stiffness
$\lambda^4 = k/4EI$
q is the general applied load
F_i is Hetenyi's functions (eqn. (4))
R_i, s_i } is the bearing reaction R at position $x = s$
W_i, d_i } is the point load W at position $x = d$
M_i, c_i } is the applied couple M at position $x = c$
w_i, a_i, b_i } is the uniformly distributed load w from $x = a$ to $x = b$

Notation used only in Appendix A

v_i is the vector of primary unkowns, deflection, slope, bending moment, and shear force (eqn. (A.1))
a_{ij}, b_i } relate general v_i vector to values at start of beam (eqn. (A.2))
u_i is the notional vector of two current unknowns
t_{ij}, k_i } relate v_{0i} vector at start of region to u_i vector

These quantities when barred refer to general v_i vector.

Introduction

The problems associated with a beam resting on an elastic medium have been studied for many years, although the primary aim has been to study the behaviour of a beam resting on soil. The basic Winkler formulation[1], in which the soil is assumed to provide a reaction in proportion to the deformation imposed on it, has been applied to many problems (see, for example, Hetenyi's classic book[2]) and is known to give good agreement with more exact formulations for the sub-grade reaction[3].

Pavlovic & Tsikkos[4] used the terminology quasi-Winkler to classify foundations which provide a reaction only when the deflection is downwards; that definition will be retained in this paper. When the deflection is upwards, the beam is assumed to lift off the foundation, and there is no force between the two elements. Pavlovic concluded that, for a downwards point load on the beam, lift-off occurs some way from the load itself and the maximum bending moment in the beam differs little from the exact Winkler model. There were significant differences between the two models when the beam was loaded by a couple, since the beam lifts off the ground on one side of the applied load. This would be a relatively rare case in practice for soil-structure interaction problems, since structural elements that apply moments to a foundation, such as columns, normally also apply a point load that would be sufficient to force the beam back on to the soil, at least in the vicinity of the column.

However, there is a class of problem for which these effects should be studied more closely. Post-tensioned prestressed concrete beams are built, unstressed, on falsework. They may be either cast *in situ* or precast in segments and assembled on site. The cables are then progressively tensioned until the beam lifts off the falsework; when the beam is completely free, the temporary supports can be removed. In theory, the falsework can be removed at a slightly earlier stage, when the beam has sufficient prestress to support its own weight, but in practice the stressing is continued until separation occurs.

It is important to be able to calculate the forces in both the beam and the falsework throughout the stressing procedure. Any stress changes in the beam after a cable has been anchored will cause a change in the prestress (usually a loss), so we need to be able to determine the stress at the time of transfer for every cable. In practice, this means that the bending moments in the beam at all stages of the stressing process must be determined.

A large beam may have many cables, with varying profiles. Normally, the total effect of these cables is to raise the whole beam, but individual cables may cause downward deflections over certain areas of the beam. These can give rise to very high local reactions from the falsework, which may cause problems for the beam itself, for the falsework, or for the falsework foundations.

We thus need to be able to determine the variations in the beam bending moment, the loads carried by the temporary supports, and the reactions transmitted to the permanent supports.

Definition of problem

For any given cable profile, or set of profiles, we can calculate the forces that the tendon exerts on the concrete. These forces will in general comprise distributed loads where the cable is following a curved profile, together with point loads and moments at the anchorage positions. The determination of these loads can be regarded as a separate preliminary exercise whose details are already well established[5]; they will therefore not be considered here.

The falsework supporting the prestressed beam will normally have a much lower stiffness than the beam's permanent supports, and if the structure tends to deflect downwards they will provide a positive reaction. However, if the beam deflects upwards, as it will eventually do over the majority of its length as sufficient cables are stressed, there will be no contact between the beam and falsework, and hence no reaction between them. The falsework can thus be considered to be a quasi-Winkler foundation.

Our problem can thus be regarded as one in which we wish to analyse a beam, continuous over a number of supports (the permanent bearings), supported elsewhere on a quasi-Winkler foundation and subjected to a combination of point loads, distributed loads, and moments.

We shall define some nomenclature to avoid confusion later on:

Permanent supports, which are deemed to be rigid when subjected to vertical loads but provide no restraint against rotation, will be termed *bearings*, to distinguish them from the reactions from the falsework, which will be termed *supports*.

A *Winkler* support will be one which provides an elastic restraint against both positive and negative displacement, while a *quasi-Winkler* support will restrain only positive (downward) deflections.

Possible solution strategies

Various solution strategies can be considered.

(1) The analytical technique of Pavlovic & Tsikkos[4]. The form of the displaced shape must be known, so that the positions of the loads and bearings relative to the interface between the supported and unsupported regions are known, at least to the extent of being able to specify whether they are to the left or right. It is not necessary to specify the position of the interface precisely.

The displaced shape is divided into regions, each being *either* supported *or* unsupported. The form of the governing equation in each region is known, and the displaced shape in that region can be specified in terms of four parameters. The positions of all of the interfaces between the regions are also variable parameters of the problem.

By specifying conditions of compatibility and equilibrium between adjacent regions, and also that the displacement is zero at the interfaces, it is possible to set up sufficient non-linear simultaneous equations to be able to determine all the unknown parameters. It should be noted that the non-linearity relates only to the position of the interfaces. If the interfaces are fixed, the equations become linear, a fact which we shall make use of later.

The drawback of the method as presented is the difficulty of determining *a priori* the form of the displaced shape, which means that each problem has to be set up afresh; we are trying to produce a generalised solution technique.

(2) The beam can be analysed as an ordinary beam, supported only on its permanent bearings, under the action of the applied loads. In areas where the displacement is positive, we can determine the corresponding load that the support would provide, and reanalyse the structure under the combined effects of the applied and support loads. This will result in a new displaced shape, for which we can reassess the support loads and repeat the process.

In theory, the process should converge to a displaced shape which causes support loads that are in equilibrium with the applied loads, but in practice there are a number of problems.

The displaced shape will, in general, be varying continuously, so the support loads will also vary in the same way. Since we cannot analyse a beam under a completely general loading, we must idealise the support loads as a series of discrete loads, either by a large number of point loads or, if we wish to be more sophisticated, as a series of uniformly or trapezoidally distributed loads. Whichever method we use, we are introducing an error and a considerable degree of complexity.

There is a more serious problem, however. Unless the support stiffness is very low, the load applied by the support after the first iteration will be sufficient to lift the beam off the support for the second iteration. The beam will thus oscillate, alternating between positive displacements that cause support loads and negative displacements that do not. Convergence occurs only if a proportion of the change in the support load is applied to the member; the determination of the proportion to be used is critical, since if it is chosen too small, convergence is very slow and, if too large, oscillation occurs.

This method is not suitable for use in a generalised program.

(3) A better alternative, which can be regarded as the opposite of the previous method, is to treat the beam as wholly supported on a Winkler foundation, but to apply 'fictitious' loads where the beam is off the foundation, to counteract the negative reaction produced by the assumed elastic support.

This method does not suffer from the oscillation problem of method (2), in that the beam does not alternate between positive and negative displacements. It does, however, tend to oscillate about the final solution, and convergence is slow; this is compounded by the fact that the individual beam analysis calculations are considerably more complex than for a normal beam.

The other problem identified above, associated with the idealisation of the support reactions, still remains and adds to the difficulty of providing a reliable generalised package.

A program has been written to analyse beams by this method, but convergence was found to be extremely slow and the number of fictitious loads needed to correctly model the regions where the beam lifted off was unacceptably high[6].

(4) The final method presented, which is the one described in more detail below, can be regarded as a generalisation of Pavlovic's method, but avoids the difficulties associated with non-linear equations.

The beam is assumed to be divided into regions. Every alternate region is deemed to be supported on Winkler foundations (not quasi-Winkler), but the ones in between are not. Such a beam can be analysed relatively easily by a combination of the classical Macauley's method for the unsupported beam and Hetenyi's method for the Winkler supported regions. Permanent bearings can be incorporated without difficulty. The equations to be solved are all linear.

The results of this analysis will be a deflected shape with regions of both positive and negative deflection. These regions can be identified and used to define the support conditions for a repeated anlaysis. Regions where the beam is deflecting downwards will be assumed to be on a Winkler support for the next analysis; where it is deflecting upwards, it will be assumed to be unsupported.

After a few iterations (typically, 5 or 6 in the cases studied by the authors), there are no changes in the displaced shape, so the support conditions assumed correspond to a quasi-Winkler foundation. The solution will be exact, since no approximations are being introduced by assuming the existence of fictitious discrete forces.

Details of method

The technique adopted for the solution of our problem will be the 'method of initial parameters' identified by Hetenyi[2], extended to include generalised loading and adapted to cover normal beams as well. For completeness the method will be summarised here, together with the equivalent terms for the normal beam regions.

Consider the Winkler region shown in Fig 1. The displacement y is governed by the differential equation

$$EI(d^4y/dx^4) = -ky + q \quad \ldots . (1)$$

where k is the stiffness of the elastic support and q is the applied load. The general solution of eqn. (1) for an unloaded beam is of the form

$$y = e^{\lambda x}(C_1\cos\lambda x + C_2\sin\lambda x) + e^{-\lambda x}(C_3\cos\lambda x + C_4\sin\lambda x) \ldots . (2)$$

where $\lambda^4 = k/4EI$ and C_1, C_2, C_3 and C_4 are determined from the boundary conditions at the ends of the beam; the extension of this process for a generalised loading is complicated.

It is possible, however, to relate the constants C_i to physical quantities at the beginning of the region, from which it can be shown that the shape of a beam in an unloaded region is given by

$$y = y_0F_1(\lambda x) + (1/\lambda)\theta_0F_2(\lambda x) - (1/\lambda^2EI)M_0F_3(\lambda x) - (1/\lambda^3EI)Q_0F_4(\lambda x) \quad \ldots .(3)$$

where

$$\begin{aligned} F_1(\lambda x) &= \cosh(\lambda x).\cos(\lambda x) \\ F_2(\lambda x) &= 0.5*(\cosh(\lambda x).\sin(\lambda x) + \sinh(\lambda x).\cos(\lambda x)) \\ F_3(\lambda x) &= 0.5*(\sinh(\lambda x).\sin(\lambda x)) \\ F_4(\lambda x) &= 0.25*(\cosh(\lambda x).\sin(\lambda x) - \sinh(\lambda x).\cos(\lambda x)) \end{aligned} \quad \ldots .(4)$$

These functions are shown graphically in Fig 2 and obey the following relationships

$$\begin{aligned} (dF_1/dx) &= -4\lambda F_4 \\ (dF_2/dx) &= \lambda F_1 \\ (dF_3/dx) &= \lambda F_2 \\ (dF_4/dx) &= \lambda F_3 \end{aligned} \quad \ldots . (5)$$

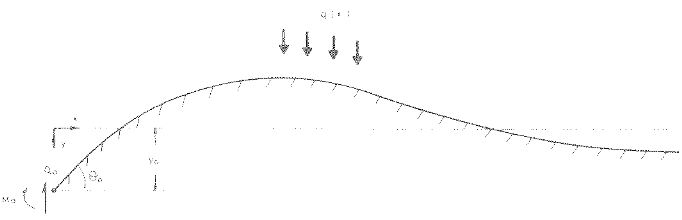

Fig 1. Winkler foundation

Note, also, that $F_1(0) = 1\cdot0$, while $F_2(0)$, $F_3(0)$ and $F_4(0)$ are all zero.

Eqn. (3) is valid only in unloaded regions, but we can generalise the result to a loaded beam.

The effect of a point load applied at an intermediate position within the region can be regarded as having a modifying effect on the beam to the right of the point of application of the load similar to a change in the initial parameters.

Thus, by analogy with eqn. (3), we can see that the effect of a concentrated load W applied at a point $x = d$, will be to modify eqn. (3) by an additional term

$$y = \ldots\ldots + W.(1/\lambda^3 EI).F_4\left\{\lambda(x-d)\right\} \quad \text{for } x > d$$

while it has no effect on the equation for $x < d$. (The load will have an effect on the *beam* everywhere and this will automatically be taken into account when the initial parameters are determined by satisfying the boundary conditions.) Similar changes are made to the expressions for slope, moment, and shear.

This idea is familiar from analysis of normal beams by Macauley's method[7], and we can simplify the equations for this work by making use of Macauley's notation, which specifies that terms contained within curly brackets $\{\ldots\}$ take their normal value if the contents of the bracket are positive and are zero if the contents are negative.

This notation will be used freely here, and we shall also use it when referring to the parameters of the F_i expressions, so that

$$F_i\left\{.-ve.\right\} = F_i(0)$$

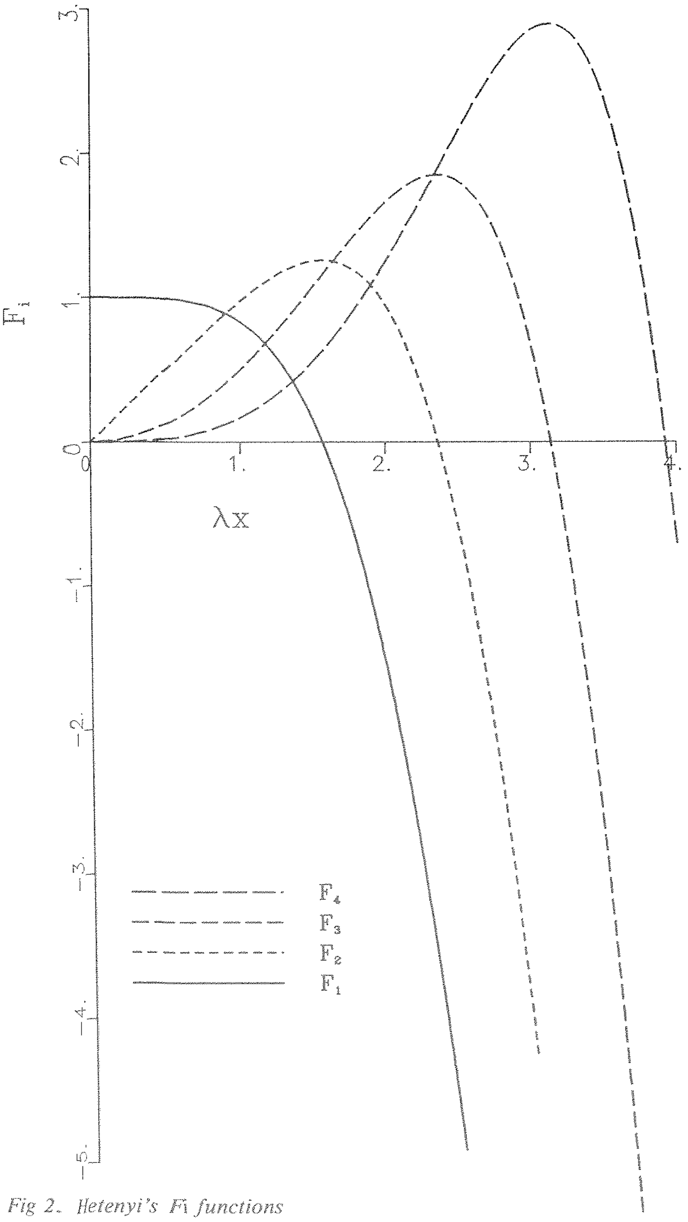

Fig 2. Hetenyi's F_i functions

These principles have been applied to point loads, distributed loads, and moments, and could be extended to other load types (such as triangularly distributed loads) if required.

The complete set of these terms is shown in Fig 3, together with the corresponding expressions for a normal beam.

We shall also need expressions for the slope, bending moment, and shear force, within each region. These are also given in Fig 3 and in Fig 4, in the same format as the expressions for the displacement.

We can now consider the complete analysis of our beam, which is, of course, made up of a number of regions, alternately on Winkler supports and unsupported.

The principal unknowns of our analysis will be the displacement and

	DISPLACEMENT (y)		SLOPE (θ)	
Load Term	Winkler	Normal	Winkler	Normal
Initial displ.	$y = y_o.F_1(\lambda x)$	$y = y_o$	$\theta = -y_o.4\lambda F_4(\lambda x)$	$\theta = 0$
Initial slope	$+\frac{\theta_o}{\lambda}F_2(\lambda x)$	$+\theta_o.x$	$+\theta_o.F_1(\lambda x)$	$+\theta_o$
Initial moment	$-\frac{M_o}{\lambda^2 EI}F_3(\lambda x)$	$-M_o\frac{x^2}{2EI}$	$-\frac{M_o}{\lambda EI}F_2(\lambda x)$	$-M_o\frac{x}{EI}$
Initial shear	$-\frac{Q_o}{\lambda^3 EI}.F_4(\lambda x)$	$-Q_o\frac{x^3}{6EI}$	$-\frac{Q_o}{\lambda^2 EI}F_3(\lambda x)$	$-Q_o\frac{x^2}{2EI}$
Bearing reaction	$-\frac{R_i}{\lambda^3 EI}F_4\{\lambda(x-s_i)\}$	$-R_i\frac{\{x-s_i\}^3}{6EI}$	$-\frac{R_i}{\lambda^2 EI}F_3\{\lambda(x-s_i)\}$	$-R_i\frac{\{x-s_i\}^2}{2EI}$
Point load	$+\frac{W_i}{\lambda^3 EI}.F_4\{\lambda(x-d_i)\}$	$+W_i\frac{\{x-d_i\}^3}{6EI}$	$+\frac{W_i}{\lambda^2 EI}.F_3\{\lambda(x-d_i)\}$	$+W_i\frac{\{x-d_i\}^2}{2EI}$
Couple	$-\frac{M_i}{\lambda^2 EI}.F_3\{\lambda(x-c_i)\}$	$-M_i\frac{\{x-c_i\}^2}{2EI}$	$-\frac{M_i}{\lambda EI}.F_2\{\lambda(x-c_i)\}$	$-M_i\frac{\{x-c_i\}}{EI}$
Uniformly dist. load	$+\frac{w_i}{k}[F_1\{\lambda(x-u)\}]_{u=x}^{a_i}$	$+w_i\frac{\{x-a_i\}^4}{24EI}$	$-\frac{w_i}{\lambda^3 EI}.F_4\{\lambda(x-a_i)\}$	$+w_i\frac{\{x-a_i\}^3}{6EI}$
	$-\frac{w_i}{k}[F_1\{\lambda(x-u)\}]_{u=x}^{b_i}$	$-w_i\frac{\{x-b_i\}^4}{24EI}$	$+\frac{w_i}{\lambda^3 EI}.F_4\{\lambda(x-b_i)\}$	$-w_i\frac{\{x-b_i\}^3}{6EI}$

Fig 3. Expressions for displacement and slope

	MOMENT (M)		SHEAR (Q)	
Load Term	Winkler	Normal	Winkler	Normal
Initial displ.	$M = y_o\frac{k}{\lambda^2}F_3(\lambda x)$	$M = 0$	$Q = y_o\frac{k}{\lambda}F_2(\lambda x)$	$Q = 0$
Initial slope	$+\theta_o\frac{k}{\lambda^3}F_4(\lambda x)$	$+0$	$+\theta_o\frac{k}{\lambda^2}F_3(\lambda x)$	$+0$
Initial moment	$+M_o F_1(\lambda x)$	$+M_o$	$-M_o 4\lambda F_4(\lambda x)$	$+0$
Initial shear	$+\frac{Q_o}{\lambda}.F_2(\lambda x)$	$+Q_o.x$	$+Q_o F_1(\lambda x)$	$+Q_o$
Bearing reaction	$+\frac{R_i}{\lambda}F_2\{\lambda(x-s_i)\}$	$+R_i\{x-s_i\}$	$+R_i.F_1\{\lambda(x-s_i)\}$	$+R_i\{x-s_i\}^0$
Point load	$-\frac{W_i}{\lambda}F_2\{\lambda(x-d_i)\}$	$-W_i\{x-d_i\}$	$-W_i F_1\{\lambda(x-d_i)\}$	$-W_i\{x-d_i\}^0$
Couple	$+M_i F_1\{\lambda(x-c_i)\}$	$+M_i\{x-c_i\}^0$	$-M_i 4\lambda F_4\{\lambda(x-c_i)\}$	$+0$
Uniformly dist. load	$+\frac{w_i}{\lambda^2}F_3\{\lambda(x-a_i)\}$	$-w_i\frac{\{x-a_i\}^2}{2}$	$+\frac{w_i}{\lambda}F_2\{\lambda(x-a_i)\}$	$-w_i\{x-a_i\}$
	$-\frac{w_i}{\lambda^2}F_3\{\lambda(x-b_i)\}$	$+w_i\frac{\{x-b_i\}^2}{2}$	$-\frac{w_i}{\lambda}F_2\{\lambda(x-b_i)\}$	$+w_i\{x-b_i\}$

Fig 4. Expressions for moment and shear

slope at the left-hand end of the member and the values of the reactions at the permanent bearings. Once these variables have been determined, all other quantities can be calculated.

Note that we do not regard the bending moment and shear force at the left-hand end as variables; they will be taken as zero in all cases, but suitable loads can be applied if fixed values are required. The method could easily be extended to incorporate these quantities if a more general analysis was needed.

The deflection, slope, moment, and shear throughout the first region can now be determined as linear functions of these two initial unknown parameters (displacement and slope). If the region includes a bearing, we shall need to introduce an additional unknown parameter, being the reaction at the bearing. However, one of these parameters can be expressed in terms of the others by making use of the compatibility condition that the displacement is specified at the bearing; we thus have only two independent parameters. We shall record the equation used to eliminate one of the parameters so that its value can be determined subsequently by back substitution.

At the end of the region, the final values of the displacement, slope, moment, and shear, become the initial values of those quantities for the next region, but they can still be expressed as linear functions of two parameters.

We work along the beam in this way, adding a parameter as we pass each bearing, but using the compatibility condition to eliminate one of the other parameters and passing from region to region. Eventually, we reach the right-hand end of the beam, where we can use the two equilibrium conditions that moment and shear are zero to solve for the two parameters we are using at the time.

The other parameters that were eliminated as we passed the supports can now be determined, using the equations recorded on the way.

The intitial values in each region can also be determined, so that the displacements, slopes, moments and shears can be calculated anywhere.

A fuller description of the techniques used in the analysis is given in Appendix A.

The initial choice of regions for the two alternative support cases was arbitrary, and the analysis above takes no account of the sign of the deflection, but we can now make a rational choice of the type of support. We seek those points where the deflection is zero; bearings will normally be such points and their positions are fixed, and it is relatively simple to find those points where the beam just lifts off the support since we have general expressions for the displacement.

A computer program has been written on these principles, running on a small micro-computer.

General data input facilities are provided, and the user can specify an initial support configuration. However, this initial configuration does not affect the final solution obtained, although of course the process can be speeded up if a good guess is provided. In practice, assuming one region throughout causes no difficulty, and since calculations for unsupported regions are much faster than for Winkler regions, the assumption of a single unsupported region is a good (and conveniently simple) starting point, unless it is known that the applied load is predominantly downwards.

Although in practice the beam will normally change from Winkler support to no support at each of the bearings, the program has been written in such a way that this is not forced on the analysis, and the initial guess can be entirely arbitrary.

After carrying out an analysis using the initial configuration, the program searches for points of zero deflection, determining how many different support regions there are and of which sort they are. The beam is then reanalysed using the new support configuration.

This process is repeated until the boundaries of the regions do not change by more than a specified small amount, so that the support conditions actually used correspond to the desired quasi-Winkler form.

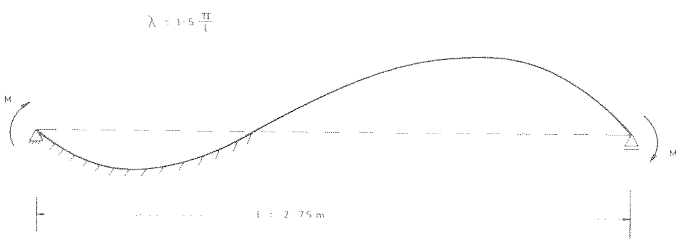

Fig 5. Beam with applied couple at each end (Pavlovic, example 2) layout

Comparison with Pavlovic

Pavlovic & Tsikkos [4] produced exact analytical solutions to two problems. The present method can be used to analyse both cases and converges to solutions identical with theirs.

To illustrate the method, Pavlovic's second example will be considered in more detail.

A simply supported beam of span l ($l = 2\cdot75$m) and with Winkler stiffness factor λ ($= 1\cdot5\,\pi/l$) is simply supported at its ends and supported elsewhere on a quasi-Winkler foundation. Point couples of magnitude M are applied in the same sense at each end, causing part of the beam to deflect downwards and part to deflect upwards (Fig 5).

For our initial configuration, we shall assume that the left-hand half of the beam is supported on a Winkler support and that the right-hand half is unsupported. This is a reasonable assumption anyway, but it is also the result that would have been produced by considerations of symmetry after one iteration if the whole beam had been assumed to be either wholly supported or wholly unsupported.

The position of the lift-off point (measured as a distance from the left-hand end) after each iteration is as follows:

Initial value	1·3750
After iteration No.	
1	0·994303
2	0·819241
3	0·774718
4	0·773520
5	0·773584
6	0·773585

Fig 6 shows the deflected shape after a number of iterations. After three iterations the deflected shape is sensibly indistinguishable from the exact result, and after six the results are identical.

Beam lifting off falsework

We can now consider a practical example. Fig 7 shows a beam of prismatic cross-section, with three unequal spans of 40, 50, and 45 m, respectively. The beam is assumed to have a second moment of area of 5 m^4, which will give a short-term flexural stiffness of 150 kN/m^2, and to have a weight of 150 kN/m. These values have been chosen arbitrarily, but are typical of those for a single-cell box beam carrying a two-lane road over a 50 m span.

The beam is to be prestressed with 20 cables which, for the sake of simplicity in calculation, we shall assume to be all following the profile shown; each carries a constant force of 5062kN at transfer. (The simplifications inherent in these assumptions will not cause the problem to differ from reality and will avoid cluttering up the problem with unnecessary detail). These cables are to be stressed in succession; the cables are detailed in such a way that the lateral forces from eight cables approximately balance the beam's dead load, but the problem has been deliberately made unsymmetrical so that lift off from the falsework occurs at different times in the different spans. The equivalent lateral loads transmitted to the concrete by one tendon are also shown in Fig 7.

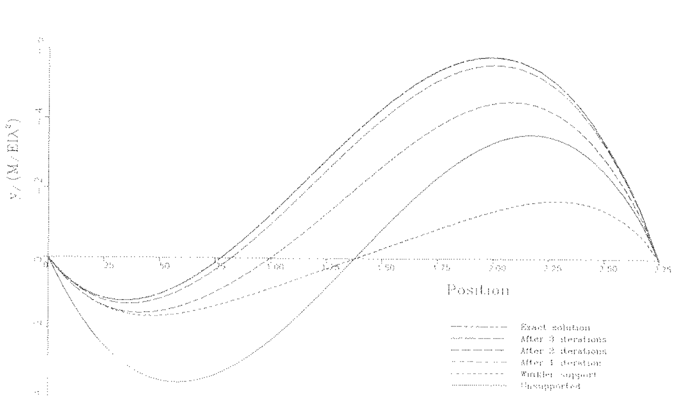

Fig 6. Beam with applied couple at each end (Pavlovic, example 2) deflections

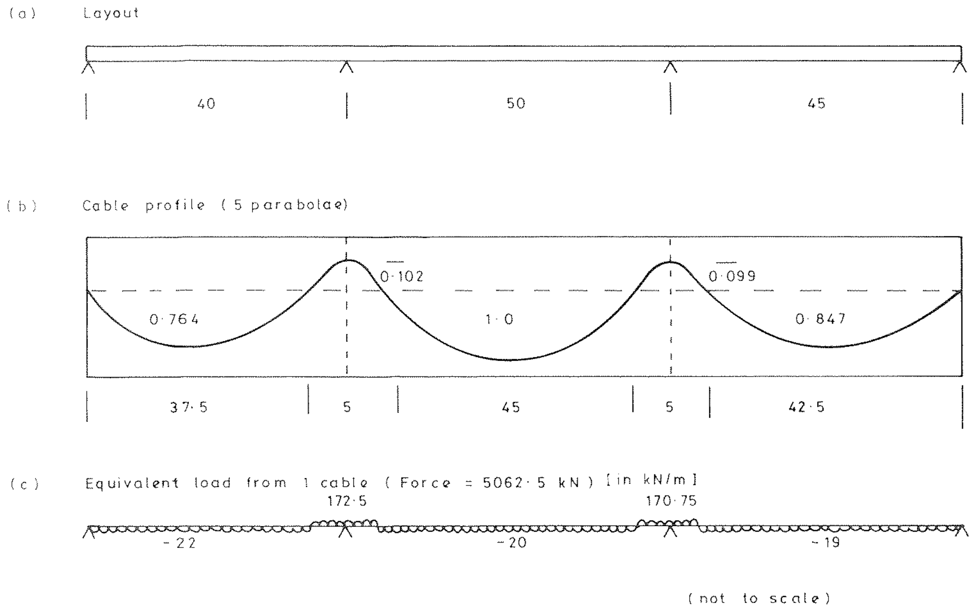

Fig 7. Three-span beam: layout, cable profiles, and equivalent loads

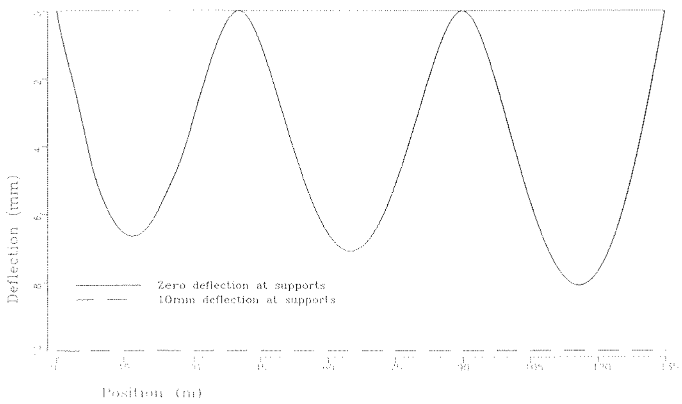

Fig 8. Three-span beam under dead load only: effect of bearing displacement on deflection

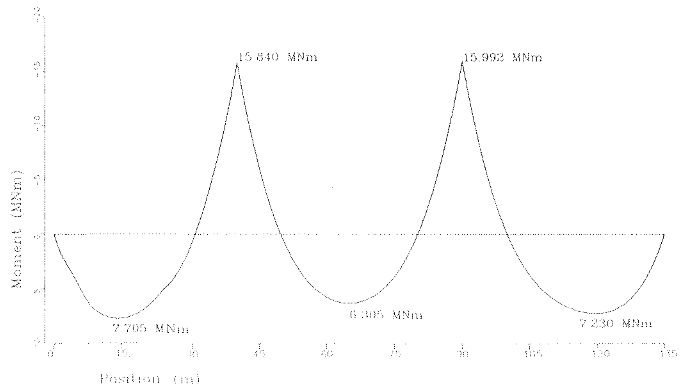

Fig 9. Three-span beam under dead load only: beam on Winkler support and non-deflecting bearings

An assumption has to be made about the stiffness of the supporting falsework. It is reasonable to assume that the falsework deflects 10 mm under the action of the beam's dead weight. This corresponds to a falsework stiffness of 15 000 kN/m^2. For simplicity, we shall refer to such a support as a '10 mm support'. If the support is more flexible than this, it will be difficult to maintain the beam's geometry accurately, but a stiffer '1 mm support', corresponding to 1 mm deflection under dead load and with a stiffness of 150 000 kN/m^2, can be imagined if the beam is being cast on stiff falsework resting on a firm foundation. Results will be presented for both the 10 mm and 1 mm support conditions.

A further complexity should be considered before a detailed analysis can be carried out. This concerns the exact position of the permanent bearings relative to the displaced position of the beam under the action of the dead load alone. If the beam is analysed for this condition (i.e. with no prestress), it will not have zero bending moment along its length if the permanent bearings are assumed to have zero displacement. Away from the bearings, the beam deflects towards the more flexible falsework: at the bearings, it cannot deflect. It thus takes up a curved shape, and this will be associated with bending moments. But the beam was cast in this configuration, and no moment can be resisted by the fresh concrete, so we must look for a more rational arrangement.

Consider our three-span beam, resting on 10 mm falsework. The falsework will deflect along its whole length by 10 mm under the weight of the fresh concrete. The beam will be straight, with no induced bending moments. At this stage, the permanent bearings do not play a part, since there is no way the wet concrete of the beam can exert a concentrated load on them. The concrete then hardens in this straight, deflected, shape. The effect is that the beam has deflected 10 mm relative to its bearings before any load is applied.

Figs 8 and 9 illustrate the importance of this point. If our three-span beam is analysed under the action of dead load only, and on the assumption that the bearings *do not* deflect, we obtain the deflected shape shown by the solid line in Fig 8 and the bending moment given in Fig 9. The moments induced are quite large, but are entirely fictitious. Alternatively, we can analyse the beam on the assumption that the bearings are displaced by 10 mm. The uniformly distributed dead load causes a uniform falsework deflection of 10 mm. No load is transmitted to the permanent bearings, since the beam deflection is exactly compatible with the fixed bearing deflection.

(There is one case when the bearings should be assumed to have zero displacement. If the beam is cast without bearings, it will deflect on the falsework as before. If it is then jacked up to place the bearings at the nominal height after the concrete has hardened, either before or after the prestress is applied, the bearings should be assumed to have zero displacement.)

We now proceed to analyse our three-span beam under three different assumptions about its support conditions, using our quasi-Winkler analysis model. Results for the deflected shape and bending moment will be given for each case as the prestressing proceeds.

(1) Beam on 10 mm support, with bearings displaced by 10 mm (Figs 10 and 11).

The unstressed beam starts straight and remains in contact with the falsework until eight cables are stressed (when the right-hand span starts to lift off). By the time 10 cables are stressed, all the spans have started to lift off, but there are still large areas of the beam in contact with the falsework when all 20 cables are stressed.

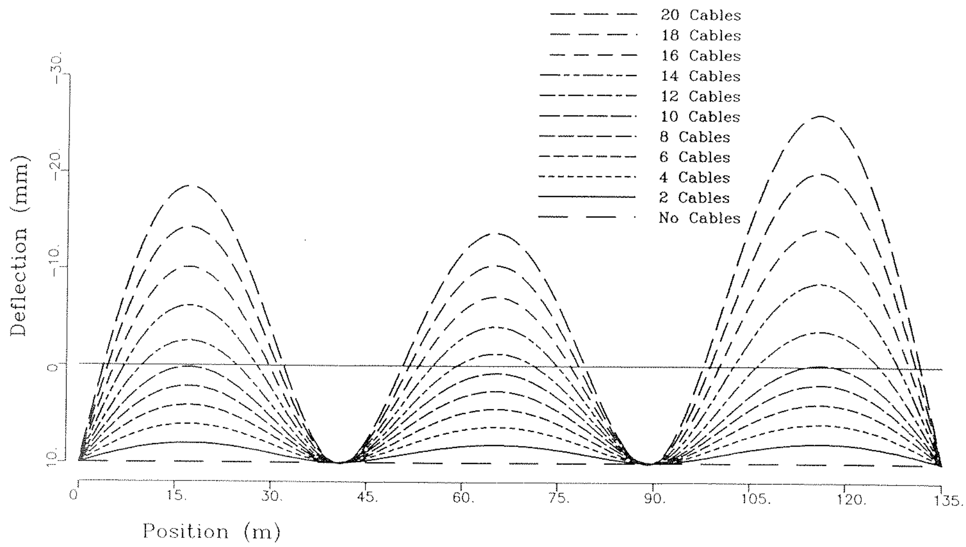

Fig 10. Three-span beam on 10 mm support, displaced bearings: deflected shape

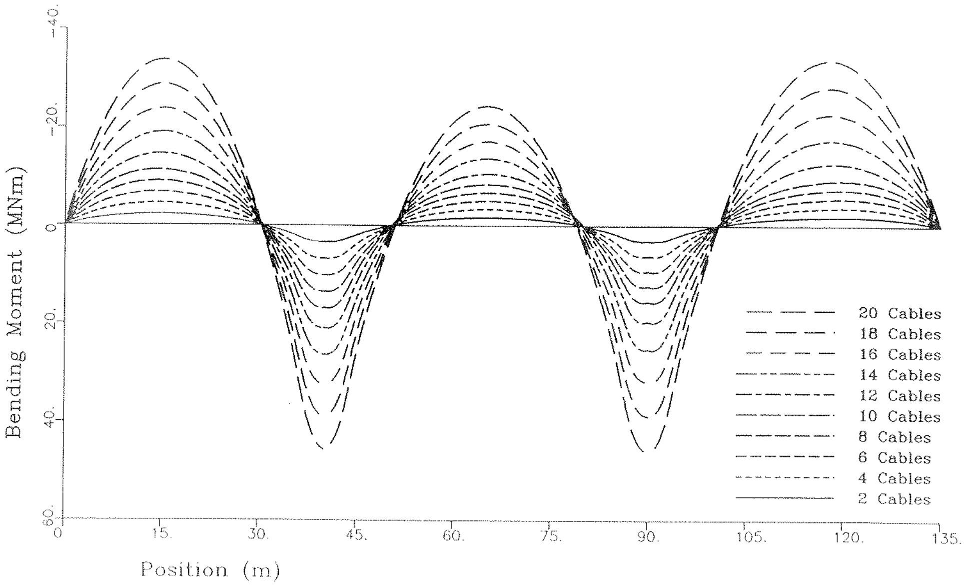

Fig 11. Three-span beam on 10 mm support, displaced bearings: bending moment

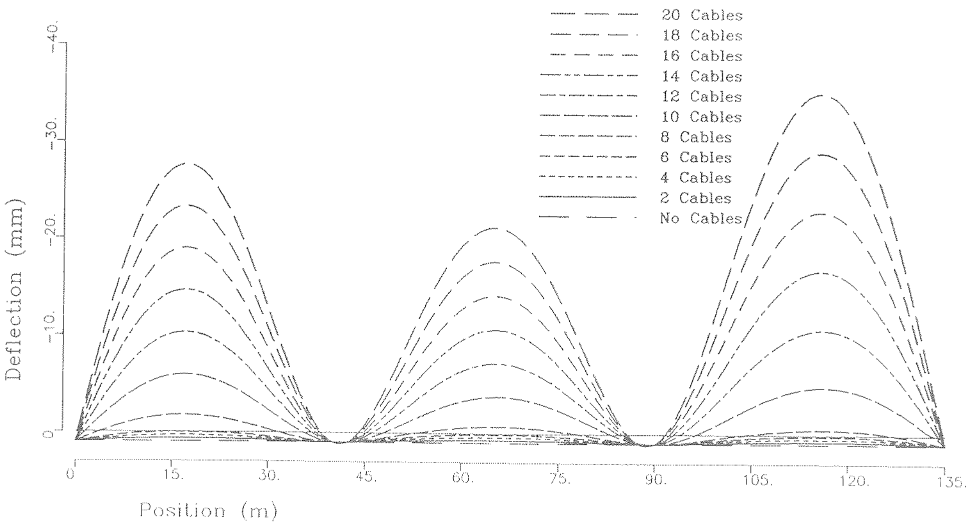

Fig 12. Three-span beam on 1 mm support, displaced bearings: deflected shape

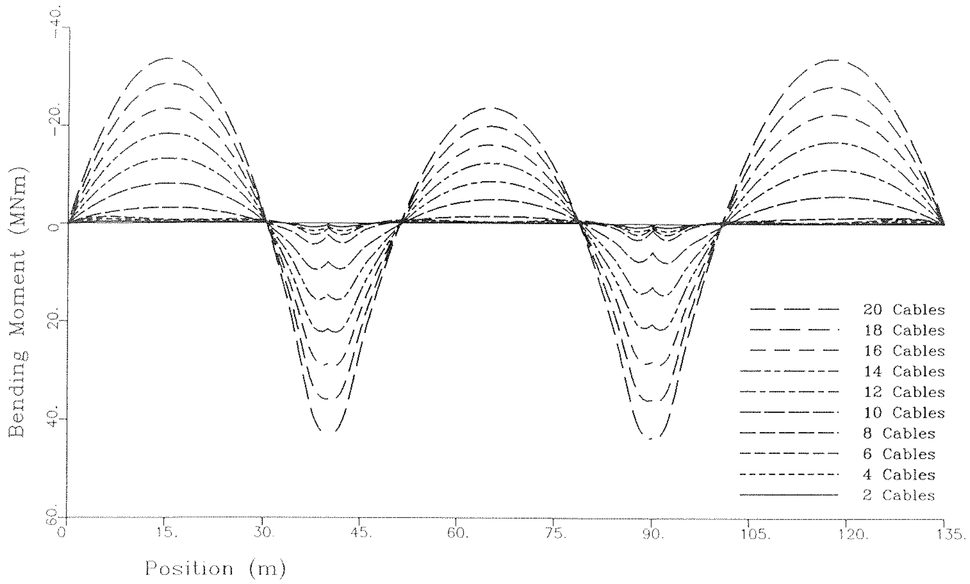

Fig 13. Three-span beam on 1 mm support, displaced bearings: bending moment

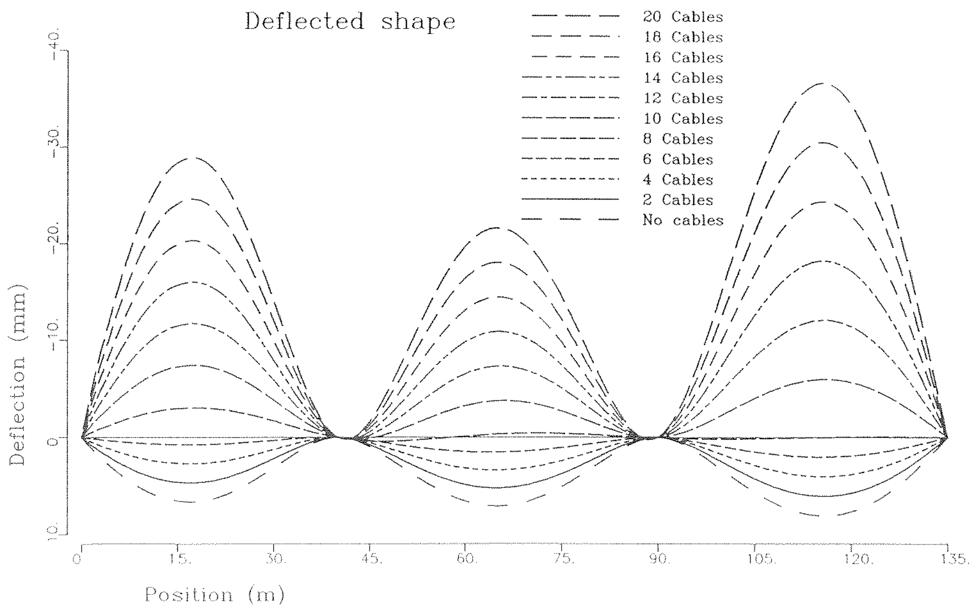

Fig 14. Three-span beam on 10 mm support, undisplaced bearings: deflected shape

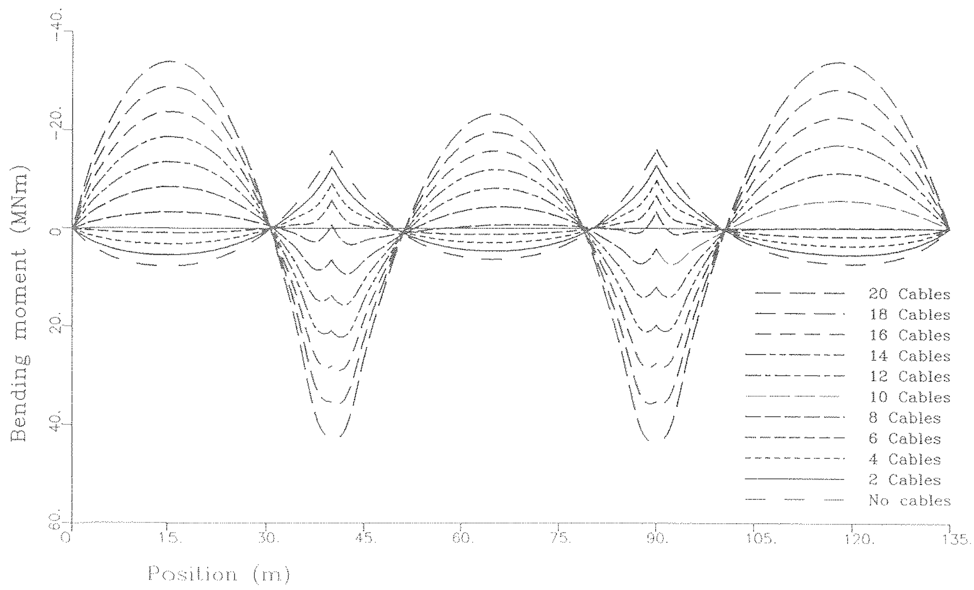

Fig 15. Three-span beam on 10 mm support, undisplaced bearings: bending moment

The bending moment in the beam starts to increase as soon as the first cables are stressed, and by the time the beam starts to lift off, there are already significant moments in the beam.

(2) Beam on 1 mm support, with bearings displaced by 1 mm (Figs 12 and 13).

The unstressed beam is straight, as in the previous case, and lift off occurs only when 10 cables are stressed. However, because the support is now 10 times as stiff, the deflections prior to lift off are much smaller than in the last case, with correspondingly lower curvatures and moments.

By the time all 20 cables are stressed, hardly any of the beam is in contact with the falsework, and the moments are virtually identical with those to be found in a beam supported only on its bearings.

(3) Beam on 10 mm support, with undisplaced bearings (Figs 14 and 15).

This is the case that corresponds to the hardened beam being jacked up to its final position so that the bearings can be placed in position before the cables are stressed. The unprestressed beam is flexed, and so has some bending moment, which of course has stress implications for the beam. The beam lifts off the shutter in all three spans when eight cables are stressed, and the beam is virtually completely free of the supports when 20 cables are stressed.

It is not easy to see the implications of this from these six plots, but by considering two specific points, we can study the phenomena in more detail. Fig 16 shows the variation in bending moment at the centre of the mainspan and over the right-hand pier as the various cables are stressed. When all the cables are stressed, there will be a hogging moment at mid span and a sagging moment over the pier.

The short dashed lines show the variation in moment, assuming that the beam is supported only on its bearings. Clearly, there is a linear variation from the unprestressed case, when the beam is loaded only by its dead weight, up to the fully prestressed condition.

When we look at the case of a beam on a 10 mm support, with displaced bearings (solid lines), the beam starts with zero moment in its unprestressed configuration, and for a while there is a linear variation of moment as successive cables are stressed. This is to be expected, since the beam is acting as though it is on a true Winkler support and behaves linearly. When the beam starts to lift off the falsework, however, the line becomes non-linear; as more of the beam lifts off, the line becomes asymptotic to the unsupported case, since the influence of the support stiffness is reduced.

This behaviour is more marked when the stiffer 1 mm support is considered (long dashed lines). The moments induced before lift off are negligible, and the subsequent response becomes asymptotic to the unsupported case much more rapidly.

The final case, that of a beam on a 10 mm support with undisplaced bearings, starts off with some moment, behaves linearly until lift-off, and then exhibits a very marked transition to unsupported behaviour.

From these results, we can clearly see that the falsework stiffness can have a marked effect on the response of the beam to the prestressing forces. Although the final response is not altered, since the falsework will be completely removed eventually, the moment in the beam at the time each individual cable is anchored can be varied quite considerably.

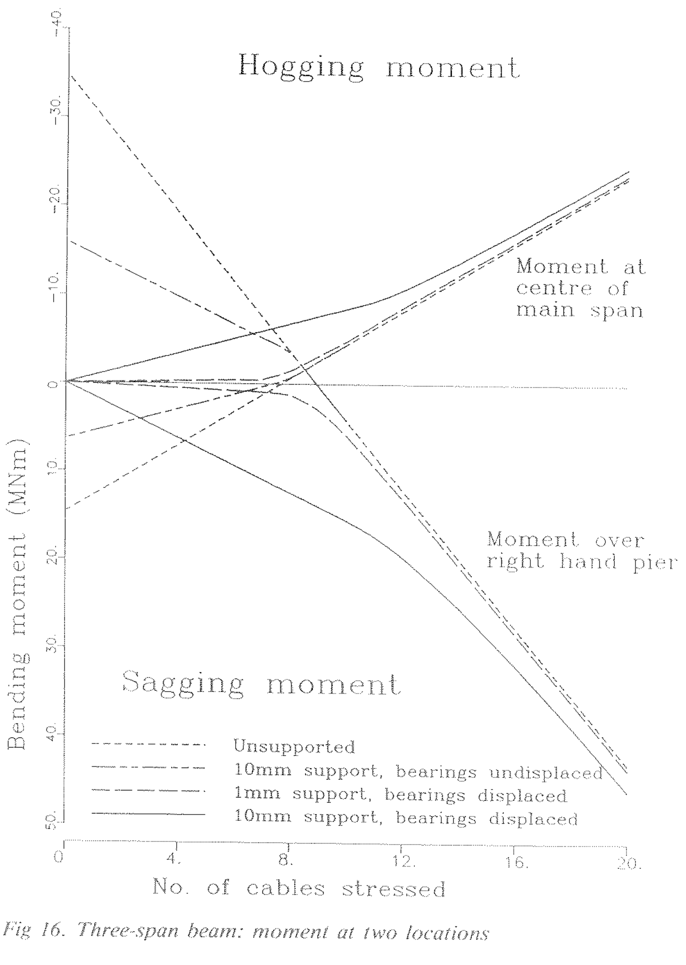

Fig 16. Three-span beam: moment at two locations

Beam loading falsework

Another problem that is difficult to analyse by classical methods is the case where the beam causes significant changes to the falsework loads, particularly when beam movements 'pinch' the falsework over short lengths.

We consider an example where a viaduct is being built on a span-by-span basis, with a significant overhang into the next span. This is commonly done to induce some dead load hogging moment over the pier and reduce the sagging moment in the middle of the recently completed span. Consider the beam shown in Fig 17; the cross-section and section properties will be the same as for the last example. A single span of 50 m with a 12 m overhang into the adjacent span will be considered, although the same phenomena will occur at each stage of the construction process.

The cable profile shown is typical of that provided during a first-stage stressing operation. It provides reasonable stresses throughout the beam when the falsework has been removed, allowing construction to proceed with the next span. Additional cables would probably be provided subsequently to carry permanent loads, but they do not concern us here.

The effect of stressing the cables is to lift the main span upwards, but this causes rotation of the beam at the pier, so the cantilever moves downwards. When the beam is analysed for its unsupported condition (i.e. when the falsework has been removed), the deflection at the tip of the cantilever is 9·6 mm downwards. It is reasonable to suppose that, when the cables are stressed, there will be some additional load on the falsework in the cantilever; it might also seem reasonable that, since the peak downward deflection in the unsupported beam is at the tip, a stiffer prop provided there would relieve the intermediate falsework of the additional load.

The beam has been analysed for two of the support conditions considered earlier for the three-span beam; Fig 18 shows the reactions in the falsework.

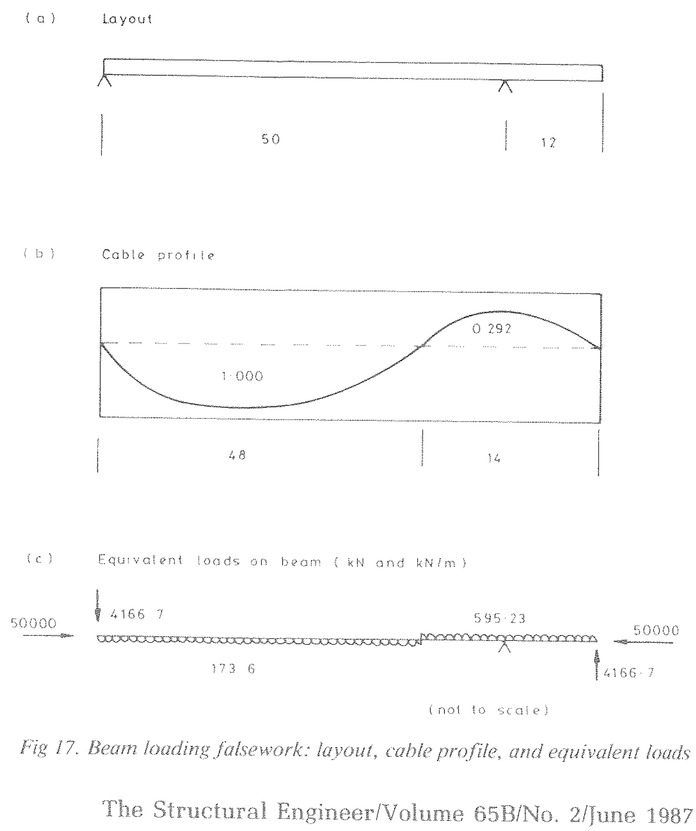

Fig 17. Beam loading falsework: layout, cable profile, and equivalent loads

For the 10 mm support, with the bearing displaced so that the unprestressed beam is not loaded, the support reaction drops off fairly rapidly at the left-hand end from the dead load value of 150kN/m, as is to be expected. The beam is free of the falsework over the central 20m or so of the main span, and then comes into contact with the falsework again, over the support and right up to the tip of the cantilever. The peak reaction on the falsework occurs about 6 m into the cantilever, with a magnitude of 173kN/m, which represents an increase of about 12 % over the dead load value for which the falsework was presumably designed. At the end of the cantilever, the reaction is considerably reduced (149kN/m).

A more significant effect is observed for the 1 mm support with deflected bearings. The beam lifts off the shutter over a larger length, but the peak intensity of the reaction on the falsework is much higher (231kN/m or 54 % over the dead load value) and is concentrated much closer to the bearing. This additional support serves also to cause the cantilever to bend upwards. Indeed, the tip of the cantilever now lifts off the falsework, so any additional propping provided at that point would not carry any load and would not relieve the falsework elsewhere.

Such an increase in the falsework loads could have serious implications for the falsework designer, given the lower factors of safety often used when designing temporary works.

Fig 19 shows the variation in bending moment in the beam for the different support conditions, under the action of dead load and prestress. There are clear differences in the bending moments between the situation when the falsework has been removed (for which the beam is presumably designed) and the cases when the falsework is still present (for which, in all probability, it has not been designed). These variations are particularly marked in the vicinity of the pier and may have a large effect on the stresses in the beam.

The effects on the stress may be large enough to consider a revision of the construction process and should not be ignored. For example, they could mean that the stressing operations may have to wait longer for the concrete to reach the required strength, or that the cable profiles need to be altered to eliminate tensile stresses which occur only at the intermediate stage when the beam is stressed but still in contact with the falsework.

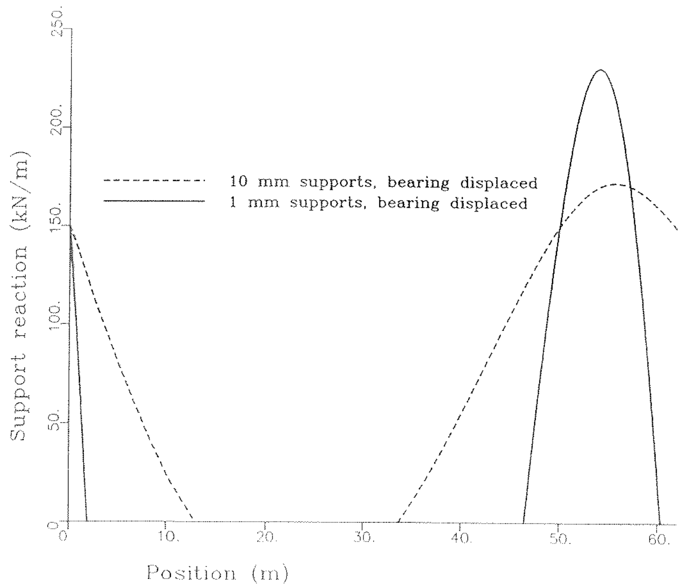

Fig 18. Beam loading falsework: support reactions

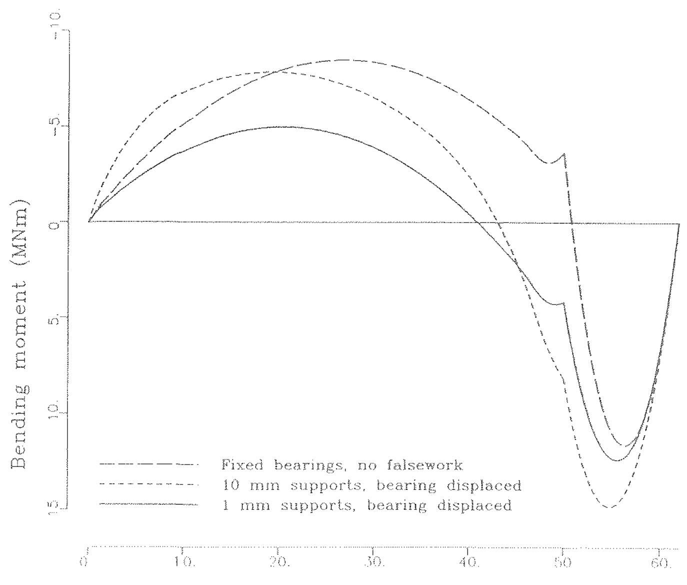

Fig 19. Beam loading falsework: bending moment

Conclusions

A method has been presented for the analysis of beams resting on supports which provide reactions if the deflection is downwards, but not if the deflection is reversed.

The method has been checked by comparison with some published analytical solutions and been shown to give rapid convergence to the exact solution.

Illustrations have been given for cases where a post-tensioned beam peels off its supporting falsework as the beam is stressed. It has been shown that the stiffness of the falsework, and the exact nature of the support conditions, have significant implications for the design of the beam itself, and also for the temporary works design.

Possible extensions to the method

The method as outlined in this paper applies to beams simply supported on a number of rigid bearings.

The quasi-Winkler support has a constant stiffness throughout the length of the member, and the permanent bearings can all be displaced by the same fixed amount. The applied loadings can consist of point loads, uniformly distributed loads, and point couples.

The method could be extended to cope with additional boundary conditions (such as fixed ends or lines of symmetry) or supports on springs restraining either vertical displacement or rotation.

The quasi-Winkler support could be divided into zones, each of which could have a different stiffness, to simulate falsework of different rigidity. In this case, the end of each of these zones would have to form a boundary between regions used in the analysis process, and allowance would have to be made for the fact that alternate regions are not necessarily unsupported.

An extension of the method to cover the case where the support stiffness varied in a more complex way would not be easy, since this would change the basic governing differential equation.

Additional loading cases (to cover, for example, trapezoidal load distributions) would not be too difficult to implement.

The method does not lend itself to beams of variable stiffness because of the difficulty of performing integrals which involve terms of (I/EI); this is a disadvantage of Macauley's method as well. Nor can its extension to grillage-type structures be readily contemplated. However, the underlying philosophy, in which the structure to be analysed is divided into those areas in contact with the support and those separated from the support, and then iterating until the regions chosen correspond to regions with positive or negative displacement, should be applicable to a wider range of grillage or finite element analysis programs.

Glossary

Support: refers to the medium that provides reactions throughout the length of the beam

Bearing: the permanent bearings that provide reactions at discrete points

Winkler support: a support that resists positive and negative displacements

Quasi-Winkler support: a support that resists positive (downwards) deflections only, but does not affect negative deflections

Unsupported: indicates that the beam is not supported continuously; it does not indicate that there are no bearings

10 mm support: indicates a support which deflects 10 mm under the action of the beam's own self-weight

1 mm support: indicates a support which deflects 1 mm under the action of the beam's own self-weight

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Appendix A. Details of analysis

This appendix presents details of the analysis procedure for a beam divided into regions that are either on Winkler supports or are unsupported. The beam may also be placed on bearings with fixed displacements. This analysis forms the core of the method presented in this paper.

The analysis is performed in two stages. In the first, the values of the primary unknowns are determined; in the second, these values are used to calculate details of the deflected shape everywhere else in the beam.

Define a vector v_i (i = 1,2,3,4)
where v_1 = y deflection
v_2 = Θ slope
v_3 = M bending movement
v_4 = Q shear force ....(A.1)
at a point (x) within the beam

In particular, define a vector v_{oi}, which are the values of the variables at the start of a region, where x = 0.

The general vector $v(x)$ can be related to the initial vector by:

$$v_i = a_{ij}\, v_{oj} + b_i \qquad \text{....(A.2)}$$

(We adopt the summation convention where repeated suffices imply summation)

The coefficients a_{ij} and b_i are functions of x and can be found easily from the expressions in Fig 3 and Fig 4; the a_{ij} relate to the initial conditions at the left-hand end of the region, and the b_i depend on any loads and fixed bearings within the region.

Thus, once the v_o values are determined, the general v quantities can be determined easily.

We do not need to treat initial values at the start of every region as unknowns (which is the essential element of Pavlovic's work). Instead, we use the fact that the values at the end of one region become the starting values of the next region. We can then express the value of v anywhere in the beam in terms of the four initial values at the extreme end of the member, and the value of the reaction at fixed bearings.

Even this number of parameters can be reduced further, since we can specify that M_o and Q_o are both zero at the start of the member. Any load applied at the end can be regarded as being applied infinitesimally adjacent to the end without loss of generality, so for a beam on n bearings, we shall have $(2+n)$ unknowns.

We need to consider only two quantities as unknowns at any one time, however. We start at the left-hand end, where y_o and Θ_o are unknown. From this point, until we reach the first support, any quantity can be expressed in terms of these two unknowns. At the first bearing, the displacement is fixed, so we can establish a relationship between y_o and Θ_o which we use to eliminate y_o as an independent variable.

Between the first and second bearings, we can express any quantity in terms of the two independent unknowns Θ_o and R_1 (the reaction at the first support). At the second bearing, we can eliminate R_1 in the same way that we eliminated y_o at the first.

The process continues in this way until we reach the extreme end of the beam, where our two independent unknowns will be R_n and Θ_o. At this point, we can solve for the two unknowns using the conditions $M=0$ and $Q=0$. We can then use the relationships already established between y_o and Θ_o, R_1 and Θ_o, etc., to determine all the unknowns.

Θ Alternative formulations based on similar principles could be adopted. It might be desirable to eliminate y_o and Θ_o at the first two bearings, to leave R_i and R_{i-1} as the current unknowns at any position, but the coding is simpler with the present method and no significant ill-conditioning problems have been encountered. If the beam formed part of a frame, it may be necessary to use other (or more) variables in the analysis.

Let us consider the solution process in more detail.

We maintain a notional vector u_i $(i=1,2)$ of the two unknown quantities. $u_2 = \Theta_o$ always, but u_1 contains successively y_o, R_1, R_2 . . . R_n.

The vector v_o at the start of any region can be expressed in terms of the current vector u_i by

$$v_{oi} = t_{ij}\, u_j + k_i \qquad \text{....(A.3)}$$

For example, at the start of the beam we have

$$\begin{bmatrix} y_o \\ \Theta_o \\ M_o \\ Q_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_o \\ \Theta_o \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{....(A.4)}$$

The vectors t_{ij} and k_i must be recomputed at the start of each region.

Elsewhere in each region, v can be found using eqn. (A.2).

$$\begin{aligned} v_i &= a_{ij}\, v_{oj} + b_i \\ &= a_{ij}\,(t_{jk}\, u_k + k_j) + b_i \\ &= (a_{ij}\, t_{jk})\, u_k + (a_{ij}\, k_j + b_i) \\ &= t_{ij}\, u_j + k_i \end{aligned} \qquad \text{....(A.5)}$$

As we pass a fixed support, however, we have to change variables, since we are eliminating one unknown and replacing it with another.

Thus, immediately to the left of a support

$$v_{li} = \overline{t}_{ij}\, u_j + \overline{k}_i \qquad \text{....(A.6)}$$

But the displacement is specified, say δ, so that

$$v_{l1} = \delta = \overline{t}_{1j}\, u_j + \overline{k}_1 \qquad \text{....(A.7)}$$

which can be rearranged to give

$$u_1 = \frac{(\delta - \overline{k}_1) - \overline{t}_{12}\, u_2}{\overline{t}_{11}}$$

This expression is stored so that old values of u_1 can be determined when u_2 is calculated.

The values to the right of the support are related to those at the left of the support, by

$$v_{ri} = v_{li} + (0,0,0,R)^T \qquad \text{....(A.8)}$$

where R is the (as yet) unknown reaction at the bearing which becomes the new primary variable.

By substituting eqn. (A.7) into the expression for v_{li}, we can obtain a revised expression for v_{ri} in terms of the new unknowns, R and Θ_0.

$$v_{ri} = \begin{bmatrix} y_r \\ {}_r \\ M_r \\ \Theta_r \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (\overline{t}_{22} - \overline{t}_{21}\,\overline{t}_{12} / \overline{t}_{11}) \\ 0 & (\overline{t}_{32} - \overline{t}_{31}\,\overline{t}_{12} / \overline{t}_{11}) \\ 1 & (\overline{t}_{42} - \overline{t}_{41}\,\overline{t}_{12} / \overline{t}_{11}) \end{bmatrix} \begin{bmatrix} R \\ \Theta_0 \end{bmatrix} + \begin{bmatrix} \delta \\ \overline{k}_2 + \overline{t}_{21}(\delta - \overline{k}_1)/\overline{t}_{11} \\ \overline{k}_3 + \overline{t}_{31}(\delta - \mathrm{k}_1)/\overline{t}_{11} \\ \overline{k}_4 + \overline{t}_{41}(\delta - \overline{k}_1)/\overline{t}_{11} \end{bmatrix} \qquad \text{....(A.9)}$$

This becomes the revised expression for v_i and defines the new value of t_{ij} and k_i.

The analysis proceeds until the two conditions that M and Q are zero at the end are used to solve for the two current unknowns R_n and Θ_0. The values of y_0 and R_i can then be determined using the equations that have been stored for u_1, to yield the complete solution.

This analysis works even if there are bearings at either or both extreme ends of the member. In these cases, a change of variable takes place immediately adjacent to the end and the correct boundary conditions are automatically incorporated.

To find the displacement, slope, etc., throughout the beam, we repeat the process, using the now known values of y_0, Θ_0 and the R_i to calculate the actual values of v_0 for each region. The values of v_i can then be inferred using eqn. (A.2).