Henderson Colloquium
Making the most of our Structures
Cambridge, July 2000

Design or analysis – elastic or plastic?

Chris Burgoyne

Dept of Engineering, University of Cambridge

We consider the relationship between elastic theory and plastic theory, design methods and analysis methods, and the upper and lower bound theorems. These raise various conflicts for engineers that can have important consequences.

Most engineers have been taught the fundamentals of plasticity theory. They know how to perform a plastic collapse analysis of a frame, or carry out a yield line analysis of a slab. They know that these give an upper bound on the collapse load and are thus "unsafe". They are aware that there is also a lower bound theorem, but circumstantial evidence shows that most engineers cannot quote it and do not believe that they use it. This is paradoxical, since they rely on it every time they design a structure.

This paradox is being thrown into relief as we spend more time assessing existing structures, rather than designing new ones. The rise of computer methods of analysis has allowed "exact" solutions to be obtained for structures that would have been far too difficult to analyse only a few years ago.

To illustrate these various aspects, we consider a hypothetical reinforced concrete slab structure. We shall assume that the structure was designed 30 years ago using the relevant methods of the day, and is now being assessed using current techniques. We will consider what today's engineers would recommend to the owners of that structure in the light of the analysis.

The structure

The structure we take as our example is a rectangular slab made from reinforced concrete, simply supported on all sides. It has an aspect ratio (μ) of 2, with the shorter sides of length a. It was designed to carry a uniformly distributed load of intensity q. The geometry is shown in Figure 1, which matches the notation used in Timoshenko [1] and other papers to which we shall refer [2-5]. It is assumed that the slab will be designed with orthogonal reinforcement. Steel in the bottom of the slab, parallel to the x-axis, resists the sagging moment M_x. Moments such as M_x are expressed as moments per unit width and in the plots will be non-

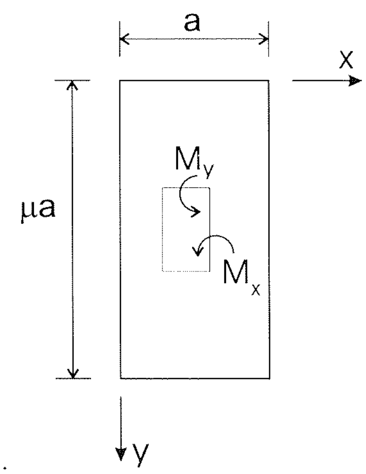


Figure 1. Slab geometry

dimensionalised by dividing by qa^2. No load factors will be applied to the structure – these mask the principles that this paper is trying to uncover. It is assumed that the structure, as with most reinforced concrete slabs, is under-reinforced.

The design

It is assumed that the structure was designed in 1970 by a bright young engineer fresh out of college. Engineers would not have had access to calculators, let alone computers, so complex calculations for a simple structure such as this would not have been justified. Our designer would be assumed to know about the methods for the elastic analysis of such slabs, such as the Fourier techniques used in the Navier analysis, but would not have had the time to sum by hand the infinite series that the solution required. But he (as it would almost certainly have been in those days) would also have known about the Hillerborg strip method of design [6] which would have been quite appealing. According to this theory, which is still taught today and would still be regarded as a perfectly reasonable method of design, the slab is imagined as a series of intersecting orthogonal strips. By apportioning the load between the two sets of strips, and designing suitable reinforcement on the assumption that the strips are simply supported beams, the designer has reduced an infinitely indeterminate plate into two statically determinate beams. The only question that our designer has to decide is the proportion of load to put onto each of the two strips. He chooses to put a proportion of the load αq onto the short strips, and the rest $(1 - \alpha)q$ onto the longer strips. He knows that whatever value of α he chooses will satisfy the lower bound theorem, which states:-

If any stress distribution throughout the structure can be found which is everywhere in equilibrium internally and balances certain external loads and at the same time does not violate the yield condition, those loads will be carried safely by the structure. [7]

The important word in this theorem is "any". Whatever value of α he picks, the solution will still be in equilibrium with the applied loads.

However, our engineer believes that it is a "good thing" to make his value of α "reasonable". He achieves this by saying that he will choose α such that the deflections in the two strips that intersect at the centre of the slab are equal (Fig. 2). This is likely to mean that the two strips will deflect together, which will in turn mean that there will be no premature cracking.

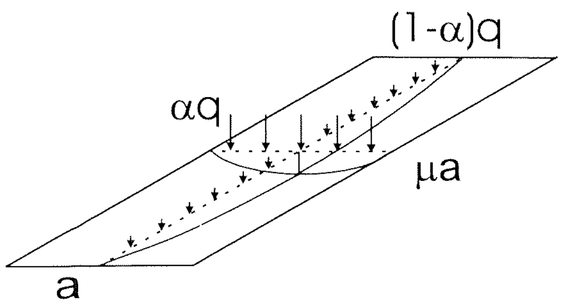


Figure 2. Choice of load distribution on Hillerborg strips.

A simple elastic calculation for a simply supported beam shows that

$$\alpha = \frac{\mu^4}{1+\mu^4}$$

so that when $\mu = 2.0$, $\alpha = 0.941$.

The engineer can then design the reinforcement in the two directions, and to be economic, he curtails the reinforcement as soon as possible, so that the resistance moment follows very closely the applied moment. He has thus satisfied the second requirement of the lower bound theorem - that the applied moment nowhere exceeds the moment of resistance.

Our designer knows that the moment field that he is designing for is not "correct", but he also knows that if the steel in one direction is overloaded, either that steel will yield or the concrete will crack more extensively, which will shed any additional load from the stiffer strips onto the less stiff loaded strips. The structure is safe.

The moment capacity fields provided to the slab in this way are shown in Figures 3 and 4.

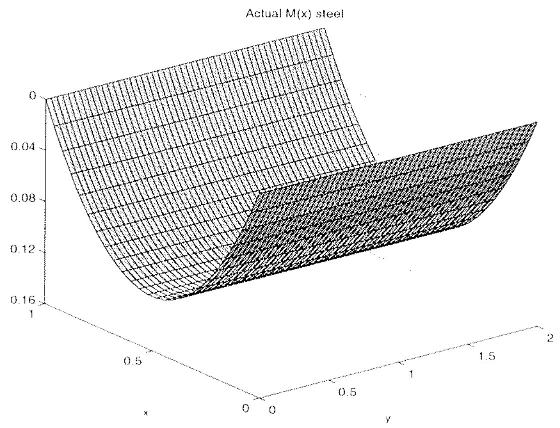


Figure 3. Actual M_x steel (max. $0.118qa^2$)

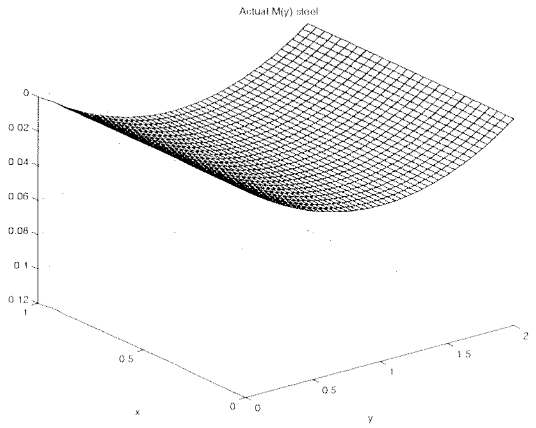


Figure 4. Actual M_y steel (max. $0.029qa^2$)

History of the slab

We continue our scenario by assuming that the structure was built to the design, using satisfactory materials and workmanship, and that it has suffered no exceptional events during its life to date. The owner of the building now wishes to have the structure checked, perhaps in preparation for a sale, or on the requirements of an insurance company, so now another engineer is approached to carry out an assessment. In the meantime, the original designers have ceased trading; the original calculations have been lost, although a set of as-built drawings have survived in the client's files. A check of the structure shows no signs of corrosion.

The assessment

The engineer who is checking the structure is also a recent college graduate. She (since times have changed) is adept at using computer analysis packages, so she carries out a finite element analysis. The slab is of uniform thickness and she has no knowledge of its state of cracking, so she takes a uniform stiffness everywhere. She takes Poisson's ratio (ν) for concrete as 0.2 and she ends up with results for three different moments, M_x, M_y, and M_{xy}, as shown in Figs 5, 6 and 7. (The figures have been derived from a Navier solution – they are what she would get if she did a finite element analysis correctly.)

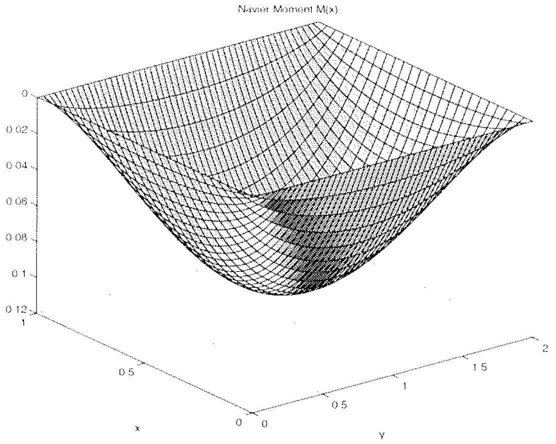


Figure 5. Elastic M_x moments (max. $0.0997qa^2$)

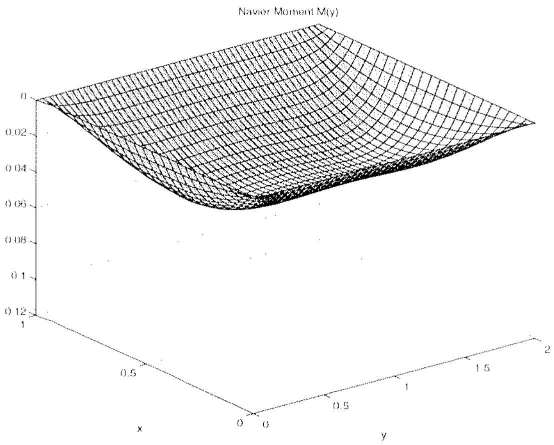


Figure 6. Elastic M_y moments (max. $0.0382qa^2$)

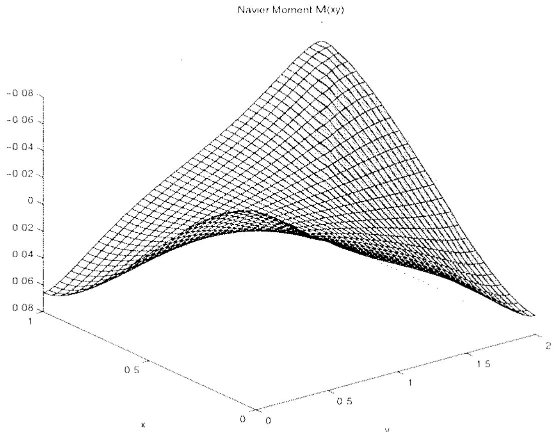
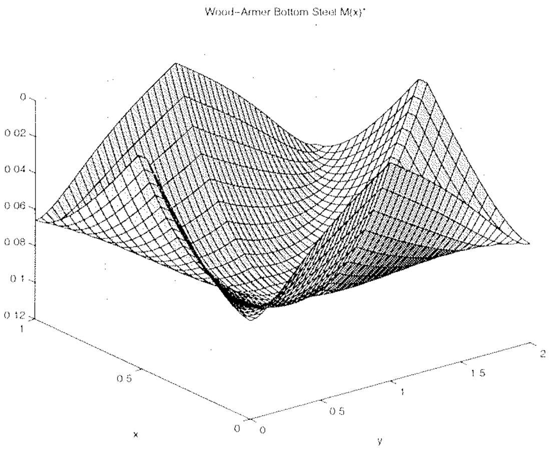


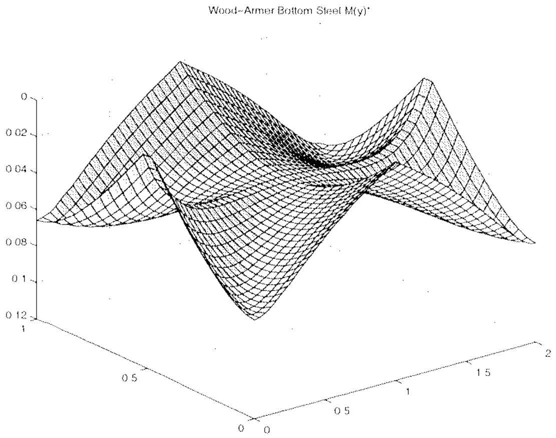
Figure 7. Elastic M_{xy} moments (max $\pm0.0659\ qa^2$)

She uses the original drawings to determine the existing moment capacities of the slab, which match those found by the original designer. The applied M_x moments are considerably less than the apparent moment capacity, but, disturbingly, the M_y moment capacities seem lower than the applied moments, and the maximum is not at the centre. There are also the M_{xy} moments to deal with. The finite element package can cope with this, however, since it has the Wood-Armer equations built-in, so she can also determine the amount of steel that the designer ought to have used.

Wood-Armer equations

The Wood-Armer results for the bottom steel are shown in Figs 8 and 9, which should be compared with Figs 3 and 4. The M_x capacity required is still less than that provided at the centre, but not at the corners; the M_y capacity is insufficent almost everywhere.



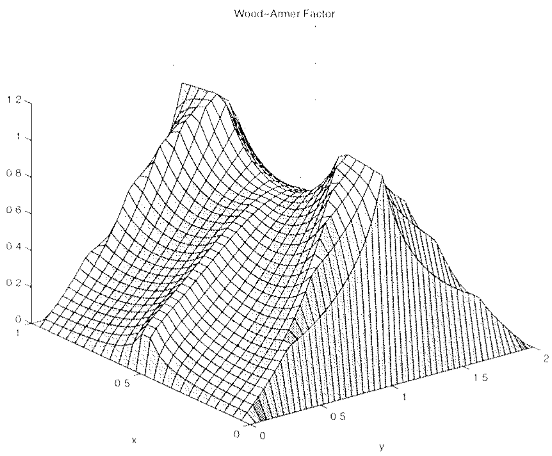


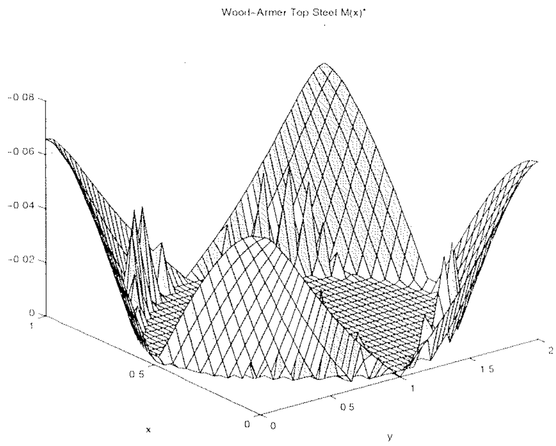
Wood-Armer results

Figure 8. M_x capacity required (max. $0.0998qa^2$)

Figure 9. M_y capacity required ($0.0401qa^2$ at centre)

Figure 10 shows the load factor that should be applied to the external load so that the bottom steel is sufficient to carry the applied moments. Values >1 are satisfactory, but this occurs only in two small regions – elsewhere the moment capacity is clearly insufficient, and by a large margin. To make matters worse, the Wood-Armer equations also show that top steel is required in the corner, to deal with the M_{xy} moments that are present, even though all of the M_x and M_y are sagging. The M_x results are shown in Figure 11.





Wood-Armer results

Figure 10. "Safe" load factor. (<1 is unsatisfactory)

Figure 11. Hogging M_x capacity required (max. $0.0659qa^2$)

Our checker knows, however, that the Wood-Armer equations were intended for use in design. By considering all possible failure orientations, Wood (for orthogonal reinforcement), and Armer (for skew reinforcement) calculated the applied moment about all possible failure orientations by simple equilibrium:-

$$M_n = M_x \cos^2\theta + M_y \sin^2\theta - 2M_{xy} \sin\theta \cos\theta$$

They then considered the moment of resistance about any possible axis (using Johansen's stepped yield criterion) in terms of the, as yet unknown, moments of resistance provided by the steel M_x* and M_y*.

$$M_n{}^* = M_x{}^* \cos^2\theta + M_y{}^* \sin^2\theta$$

There are an infinity of possible solutions which ensure that the resistance is higher than the applied moment for all orientations, but Wood added the additional requirement that the minimum amount of steel be provided. This additional requirement gives a unique solution for the required steel. A typical result is shown in Figure 12.

$$M_x{}^* = M_x + |M_{xy}| \qquad \text{and} \qquad M_y{}^* = M_y - |M_{xy}|$$

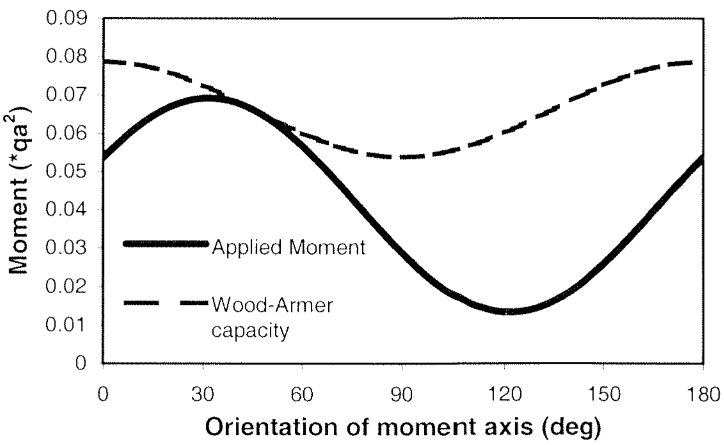


Figure 12. Moment capacity provided by the Wood-Armer equations.

Note that the capacity is always greater than the applied moment; the equations also minimise the steel required in two specified directions (here 0 and 90 degrees).

Various special cases were identified to allow for situations where these equations required moments of the wrong sign.

Denton's equations

Denton [5] recognised that structures were failing assessments because of the optimisation criterion imposed by Wood and Armer. It is of no concern to a checker that the reinforcement is not optimal. Our assessor does not have to choose the reinforcement, she only has to check whether the applied moment is less than the resistance moment for all orientations. Denton published these equations in a form that determines the limiting factor γ by which the applied moments have to be multiplied so that they lie below the resistance moment for all orientations.

So our checker applies these equations to find γ everywhere, and plots the results as shown in Fig. 13, which is in a similar format to Fig. 11. The results are better, but still show that the slab is unsatisfactory in many areas, particularly in the corners of the slab, where M_{xy} is high. γ factors as low as about 0.20 are present, and even in the middle of the slab, where M_{xy} is zero, γ is as low as 0.63.

These equations tell us nothing about the top steel, none of which was provided by our designer.

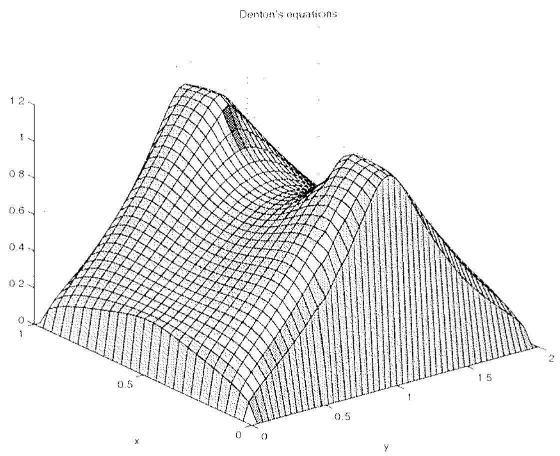


Figure 13. "Safe" load factor by Denton's equations

Our checker thus concludes that the slab is inadequate, and recommends significant refurbishment, perhaps using glued-on carbon fibre strips.

The client

The client is dissatisfied. He is informed that the structure he has been using for 30 years, without any problems and without any indication of damage, has a strength that is only a fraction of what it was designed for. He does not believe that a structure so weak would have survived without problems especially since, without admitting it to the engineers, he suspects he has been overloading it anyway. So he requests a second opinion.

Yield line analysis

As a check on the assessment, a yield line analysis is carried out. This had not been done before since our first checker knows that it is an upper bound, and thus unsafe. The second checker uses a simple yield line mechanism (Figure 14), with a single pattern parameter, which gives a load factor of 1.0 that is independent of the actual value of the pattern parameter. A more complex pattern (Figure 15),

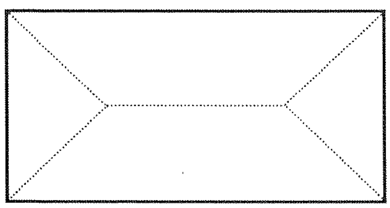


Figure 14. Simple yield-line pattern

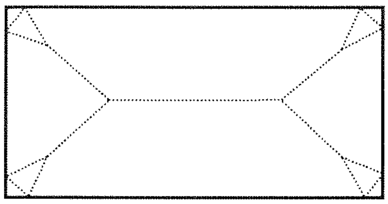


Figure 15. More complex yield-line pattern

with a hogging hinge in the corner, which ordinarily gives a lower collapse load, in this case gives a higher one. Despite extensive searching, no mechanism can be found with a collapse load factor less than one. The structure is declared safe and the client is happy, but the checking engineers feel they have lost a fruitful contract to repair a deficient structure.

Discussion

Some elements of this scenario would be familiar to most engineers working today. The example is hypothetical, and to a certain extent unrealistic. No additional safety factors have been applied which would have provided more reinforcement than was actually required. Only a single load case was considered and the reinforcement was curtailed more than would have taken place in practice. No code rules for minimum steel were applied, which would certainly have led to a slab that was stronger at its edges than is suggested here, and no suggestions for varying the proportion of load carried in the edge strips and mid-span strips were applied.

We, the external observers, are blessed with perfect knowledge, whereas our protagonists were not. Let us consider the roles of the various characters, and see whether any can be faulted, with a view to seeing whether our own procedures should be altered.

The original designer applied the logic of Hillerborg's method correctly. He may be accused of being a little simplistic in choosing a single distribution factor over the whole slab, but his attempt to equate the deflections of the intersecting strips at the centre the slab is reasonable.

Was the first checker at fault? Her reliance on computer analysis would be very typical of the procedures today. Finite element programs are cheap and easy to use – she probably spent less time setting up her analysis than the author of this paper did in writing the Fourier series solution. Did she make a mistake in using the Wood-Armer equations? Yes, but she corrected it by using Denton's equations instead. Where she was at fault was her reliance on what the plasticity community has come to call "Navier's straitjacket"[8]. This is a rigid belief that the solution produced by an elastic analysis is the true answer. The finite element package she used was linear, and had a uniform stiffness in all directions. She knew that the structure would crack, which would shed loads from one direction to another, but it is very difficult to follow the true load-deflection path since it depends so much on the history of the slab, on the yield strength of the reinforcement, on the bond characteristics and the tensile strength of the concrete, most of which are unknowable. Any elastic solution, such as the finite element or the Fourier series, gives a set of moments that is in equilibrium with the applied loads. Thus, even if it is not the correct solution, a linear elastic analysis can be used as the basis of a lower bound solution, which is ideal for design. That is the reason for the assertion at the beginning of the paper that most designers rely on the lower bound theorem every time they design a structure. They only need to ensure that they have an equilibrium set of moments and the capacity to resist them.

Could the checker reasonably have done anything else? She could have presumed that a lower bound method was used for design, but it would be difficult for her to check; there are an infinity of different ways in which the distribution factor for the loads could have been chosen. What is simple for the designer is very complex for the assessor. Knowing the reinforcement but not knowing the details of the design procedure, she could have broken the slab down into strips parallel to the reinforcement. She could have determined the loads that would just cause the moment capacity to be reached everywhere in one set of strips, and then checked whether the rest of the loads could be carried by the other set of strips. In a simple case like the one being considered here, such an approach is feasible, but in more complex cases it would be very difficult.

Should we be surprised that our upper bound method gave a load factor of 1? The original design provided just sufficient reinforcement to cope with the applied loads. Any increase in load would have caused the strips to yield all along their length in both directions. So a simple yield line mechanism, which allows all the applied loads to do positive work, and the whole of the slab to contribute to its resistance, is bound to give a load factor of 1.0. The more extensive mechanisms, where there is a corner slab in which the load does no work, are bound to give higher collapse loads. If a slab has uniform moment of resistance, corner fans can reduce the collapse load by about 10%, but here the yield lines in the corner do very little work since the moment capacity is so low.

What lessons can we learn as a profession when we are trying to make the most of our existing structures.

1. Remember the underlying structural principles, especially of the lower-bound theorem.

2. Remember that linear elastic solutions and lower bound techniques like Hillerborg, are primarily useful for design.

3. Make our structures deformable, so that the redistributions inherent in plasticity theory can take place.

4. If a structure is to fail it must have a collapse mechanism, whose collapse load can be computed.

5. A computer analysis is only as accurate as the assumptions that underlie it. Neatly printed garbage is still garbage.

References

1. Timoshenko S.P. and Woinowsky-Krieger S., *Theory of Plates and Shells*, McGraw-Hill, 1959.

2. Jaeger L.C., *Elementary Theory of Elastic Plates*, Pergamon, 1964.

3. Wood R.H., The reinforcement of slabs in accordance with a pre-determined field of moments, *Concrete*, **2**, 69-76, 1968.

4. Armer G.S.T., Discussion on reference [3], *Concrete*, **2**, 319-320, 1968.

5. Denton S.R. and Burgoyne C.J., The assessment of reinforced concrete slabs, *The Structural Engineer*, **74**, 147-152, 1996.

6. Hillerborg A., A plastic theory for the design of reinforced concrete slabs, *Procs. 6th IABSE Conference*, Stockholm, 1960.

7. Calladine C.R., *Engineering Plasticity*, Pergamon, 1969.

8. Heyman J., Navier's straitjacket, *Architectural Science Review*, **42**, 91-96, 1999.