WHY ISN'T THE MORLEY SANDWICH APPROACH TO SLAB REINFORCEMENT USED EVERY TIME?

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1 INTRODUCTION

Much of this paper was originally published by the author and Lodi in reference [1].

In order to design or assess reinforced concrete slabs it is necessary to have methods which can predict the behaviour at both the ultimate and serviceability loads accurately and, or, conservatively. For ultimate load design many methods are based on plasticity theory, for example yield line analysis [2],[3], Hillerborg's strip method [4] and the Wood-Armer equations [5]. All these methods are based on the use of moments per unit length, M_x, M_y and M_{xy}, as the stress resultants. Yield line analysis and the Wood-Armer equations both use as a yield criterion the "stepped" or normal moment yield criterion.

The normal moment yield criterion, eqn 1, was first proposed by Johansen [2] and was extensively checked experimentally by, for example, Morley [6], and Lenshow and Sozen [7] who showed that the criterion was satisfactory for reinforcement ratios of up to approximately 0.75%. Indeed it was considered that the criterion could be conservative due to the possibility of "kinking" of the reinforcement across the cracks on the tension side of a yield line.

Nielsen [8], Morley [9], Kemp [10] and others have investigated the yield criterion theoretically using the theorems of plasticity. Limits to the validity of the criterion were found. Nielsen [8], for example, investigated the capacity of slab sections under pure twisting, when the principal moments acted at + and - 45º to the reinforcement directions. It was found that for low percentages of reinforcement the yield criterion was satisfactory but as the proportion of reinforcement was increased the yield criterion became un-conservative; details are given later in this paper.

Marti et al. [11] carried out an experimental study of slabs subjected to uniform twisting moments. The slabs had the same reinforcement top and bottom, most were isotropically reinforced, the others were orthotropically reinforced. The reinforcement ratios varied from 0.25% to 1%. It was found that yield line theory over-estimated the ultimate strength by up to a factor of 2 for the higher reinforcement ratios. These results are discussed in more detail later in the paper.

Marti and Kong [12] provided an extensive theoretical study of slabs under pure twisting which confirmed the test results. May et al. [13] have also reported similar results.

In view of the evidence given above, that the yield criterion may be un-conservative, this paper describes studies using a computer based numerical model to investigate the yield criterion throughout its full range.

Morley and Gulvanessian [14] proposed that for the design of reinforcement in shell elements the six stress resultants, the three moments and the three in-plane forces can be replaced by three statically equivalent in-plane forces acting on a layer at the top of the element and three statically equivalent in-plane forces acting on a layer at the bottom of the element. The reinforcement can then be designed using the Clark [15]-Nielsen [16] equations for in-plane reinforcement, which use a yield criterion based on in-plane forces, P_x, P_y and P_{xy}. This method is described in Appendix 1 for the case of pure twisting moments.

2 THE NORMAL MOMENT YIELD CRITERION

Before discussing the results of the numerical model the normal moment yield criterion will be described and the particular case of an element under pure torsion analysed to identify the problems associated with the criterion.

The normal moment yield criterion requires that at any point within the element under consideration the moment capacity M_n^* is greater than the applied normal moment M_n for all values of θ, the orientation of the plane measured in a clockwise direction from the x axis, Fig 1.

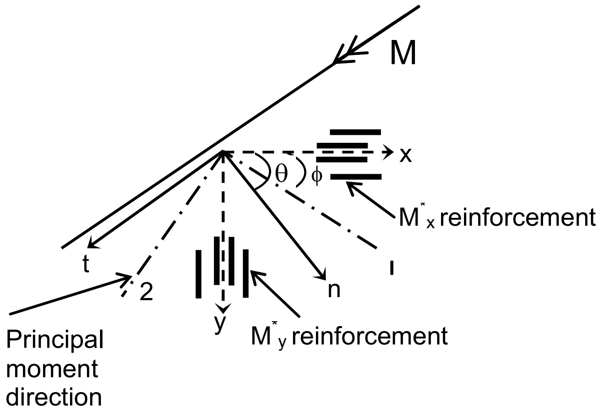


Fig. 1 Definition of axes and reinforcement directions

Thus

$$M_n^* > M_n \tag{1}$$

The yield criterion proposed by Johansen[2] is given by:

$$M_n^* = M_x^* \cos^2\theta + M_y^* \sin^2\theta \tag{2}$$

where M_n^* is the ultimate moment per unit width in direction n at θ to the x axis
M_x^* and M_y^* are the uniaxial ultimate moments per unit width in the reinforcement directions x and y respectively.

The applied normal moment per unit width is given by

$$M_n = M_x \cos^2\theta + M_y \sin^2\theta - 2M_{xy} \sin\theta \cos\theta$$

where M_n is the applied moment per unit width in direction n at θ to the x axis
M_x, M_y and M_{xy} are the applied moments in the x and y directions per unit width, respectively.

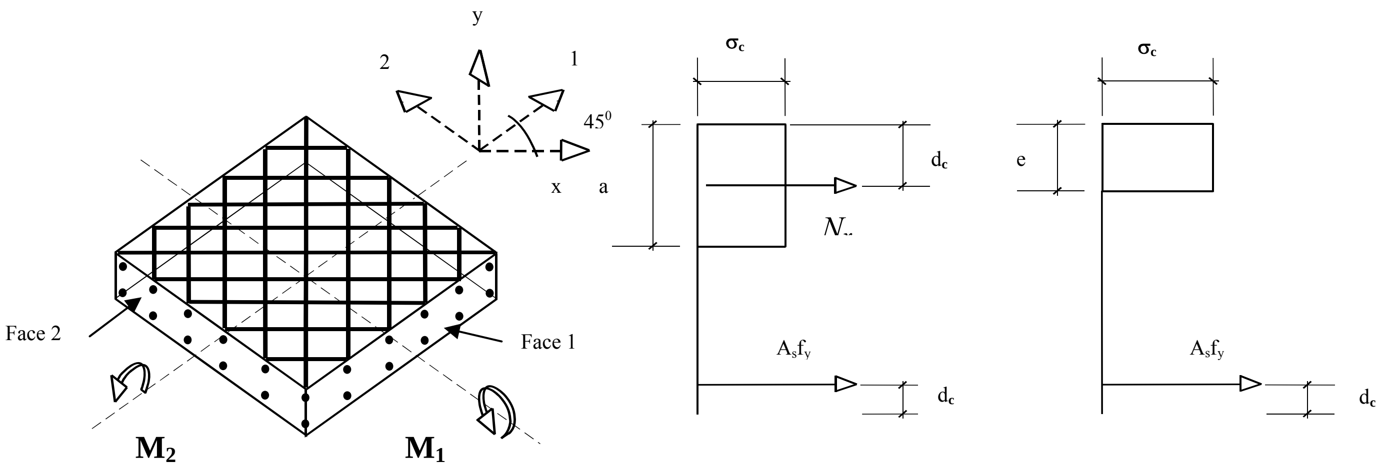
3 SLAB ELEMENTS UNDER PURE TWISTING

3.1 Exact solution

In order to identify the possible shortcomings of the yield criterion, the case of an isotropically reinforced slab element under pure twisting is considered. This is the worst case, i.e. the most unsafe for isotropic reinforcement. The principal moments are at + and - 45º to the directions of the reinforcement layers, Fig 2(a). At ultimate load all four layers of reinforcement will be at yield in tension. Assuming a rectangular stress block for concrete, Fig 2(b), then the depth of the stress block, a, will be given by

$$a = \frac{2A_s f_y}{\sigma_c} \leq \frac{h}{2} \tag{3}$$

where A_s is the area of the reinforcement,
f_y is the yield strength of the reinforcement,
σ_c is the compressive strength of the concrete.



(a) (b) (c)

Fig. 2 Slab element loaded in torsion. (a) principal planes, (b) stress distribution for ultimate moments in the principal directions, (c) stress distribution for ultimate moments in reinforcement directions.

The ultimate moments per unit width in the principal directions, M_1^* and M_2^*, are given by

$$\begin{matrix} M_1^* \\ M_2^* \end{matrix} = \pm\sigma_c a\left(\frac{h}{2}-\frac{a}{2}\right) = \pm A_s f_y (h-a)$$

where h is the depth of the section.

Knowing the principal moments, the ultimate twisting moment per unit width can be determined, which is

$$M_{xy}^* = M_1^* = A_s f_y\,(h - a) \qquad (4)$$

The maximum value of M_{xy}^* will be $A_s f_y\, h/2$ or $\sigma_c h^2/8$.

3.2 Solution using normal moment yield criterion.

To determine the twisting capacity using the yield criterion, eqns 1 and 2, requires that the ultimate moments in the reinforcement directions should be determined, Fig 2(c). The depth of the stress block, e, is given by,

$$e = \frac{A_s f_y}{\sigma_c} = \frac{a}{2} \qquad (5)$$

which is half that determined assuming the section is in pure twisting, eqn 3. This and the subsequent prediction of the ultimate moment is conservative in that it ignores the contribution of the reinforcement near the compression face of the element, however the errors are relatively small.

The uniaxial ultimate moments, and hence the ultimate twisting moment, since θ = 45º, equations 2 and 3, are then given by

$$M_{xy}^* = M_x^* = M_y^* = A_s f_y\,(h - d_c - e/2) = A_s f_y\,(h - d_c - a/4) \qquad (6)$$

If d_c, the distance to the centroid of the reinforcement, is taken as zero, since it is often small compared with h and a, then if the depth of the stress block is limited to h/2 then the maximum twisting moment $M_{xy}^* = A_s f_y\,(3h/4)$ or $3\sigma_c h^2/8$. It can be seen that the prediction of the yield criterion, eqn. 6, becomes increasingly un-conservative when compared with the exact solution, eqn. 4, as the area of reinforcement increases and hence a increases. Figure 3 shows a comparison of the non-dimensional twisting moment, $M_{xy}^{*}/\ \sigma_c h^2$ calculated using equations 4 and 6 against the ratio of the non-dimensionalised area of reinforcement, A_s/h, moment for a strength ratio $f_y/\sigma_c = 20$. It can be seen that equation 6 is increasingly un-conservative compared to equation 4. Even the prediction of equation 4 is likely to be un-conservative because the concrete is in a state of compression in one principal direction and tensile strain in the other principal direction. Vecchio and Collins [17] have shown, Fig. 4, that for concrete under this stress state the ultimate compressive strength can be reduced to about half its uniaxial strength, and that it also has a reduced stiffness. Hence the value of σ_c should be reduced to take this effect into account.

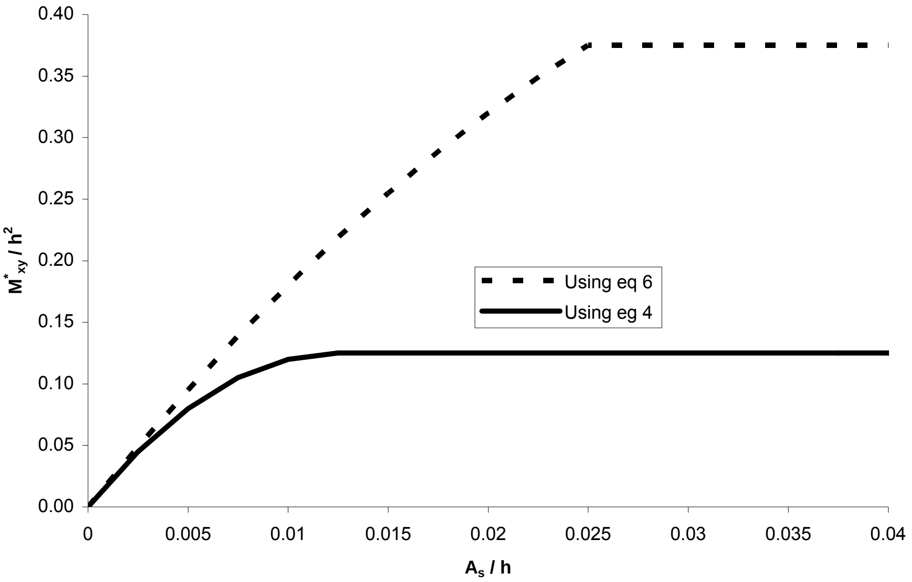


Fig. 3 Comparison of strengths given by equations 4 and 6

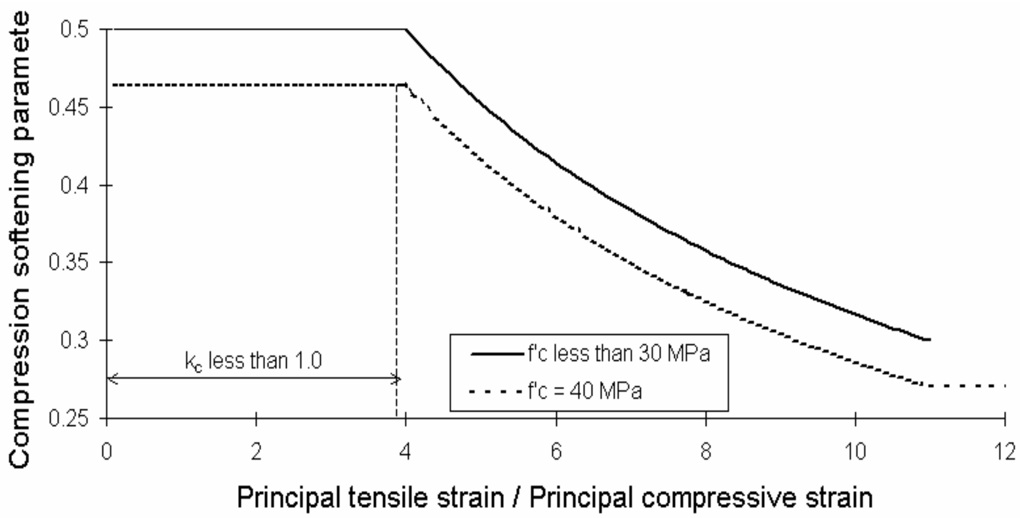


Fig. 4 Relationship between maximum compressive stress and transverse tensile strain[17].

3.3 Morley and Gulvanessian's sandwich approach

It is possible to use the alternative approach described in Appendix 1, which is a special case of the approach proposed by Morley and Gulvanessian [14]. This approach is normally applied to shell elements with the six stress resultants, moments M_x, M_y, M_{xy}, and in-plane forces P_x, P_y, P_{xy}. The forces are replaced by two sets of statically equivalent in-plane forces, N_x, N_y, N_{xy}, acting on two layers one at the top and one at the bottom of the element. In order to investigate the case of pure twisting the twisting moment applied is replaced by equal and opposite in-plane shears N_{xy} at a distance apart of h-a.

Thus

$$M_{xy} = N_{xy}(h-a)$$

and from equations A4 and A1 for isotropic reinforcement,

$$N_{xy} = A_s f_y$$

Thus the Morley and Gulvanessian approach gives identical results to equation 4.

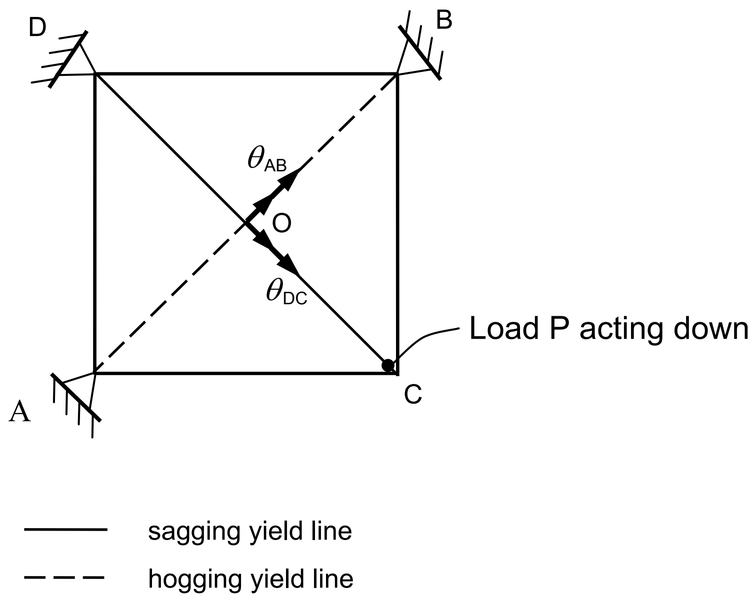
It should also be noted that the approach would give similar, but slightly smaller, uniaxial moment capacities M_x^* and M_y^* to those obtained using equation 6. The reason that the moments will be slightly smaller is that the depth of the layers is normally taken as twice the cover plus the bar diameter thus limiting the depth of the concrete stress block. Morley and Gulvanessian make a suggestion to allow the use of the concrete core to carry additional stress but in general the inclusion does not make a significant difference.

3.4 Yield line approach

In order to carry out a yield line analysis of a slab under pure twisting, the slab with supports and loading shown in Fig 5 is considered together with the square yield criterion. The point load P will give rise to a twisting moment of M_{xy} = P/2 throughout the slab. Initially a collapse mechanism comprising either the sagging yield line DC or the hogging yield line AB, each at 45º to the direction of the reinforcement, is analysed. To satisfy the normality condition the hinge capacity will be + or – M respectively and the collapse load will be given by P/2 = M_{xy} = M. If now the mechanism is considered to comprise the two yield lines, the rotations of which are taken to be of opposite sign and equal magnitude, then again the hinge capacity will be + or – M respectively. However to satisfy the normality condition at O the rotation vectors shown must be permitted. This is satisfied at point R on the square yield criterion thus the moments on each yield line are + and – M respectively. As previously the collapse load will be P/2 = M_{xy}= M.

Should the criterion be as that indicated by the dashed line, and annotated “numerically derived criterion”, the only position at which the normality condition is satisfied when the yield lines intersect is at Q, leading to a lower yield moment. Elsewhere the yield lines are still carrying a yield moment of + or – M. The work done at the intersection of the yield lines will obviously be less per unit length of yield line than with the square yield criterion. But, as the length of the yield lines at the intersection is zero the work done at the intersection is insignificant so the collapse load remains as for the square yield criterion. Thus again the collapse load will be P/2 = M_{xy}= M.

If the number of yield lines is increased such that the entire slab yields simultaneously in positive and negative bending, then, as previously, the only position at which normality condition is satisfied is Q, Fig 5, but now throughout the entire slab. The simultaneous failure of all the material in the slab is possible because the moment M_{xy} is constant through out the slab. Thus it can be seen that the use of the square yield criterion with yield line analysis will give greater collapse loads, if there are intersecting positive and negative yield lines, than the numerically derived criterion. In a later section it will be shown that the square yield criterion becomes un-conservative as the percentage of reinforcement increases.



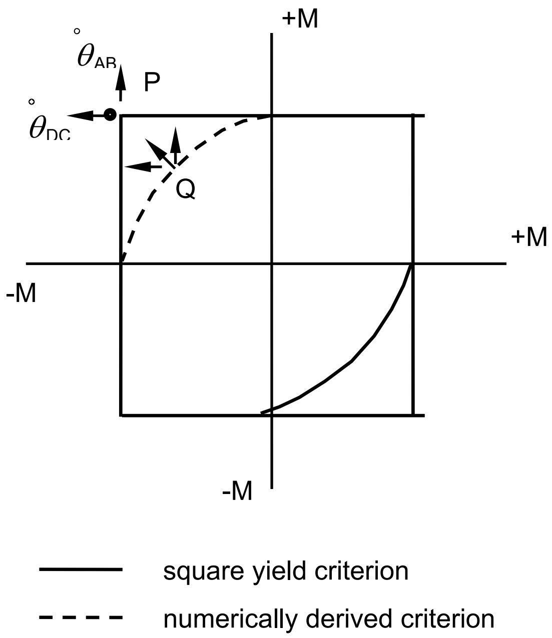


Fig. 5 Yield line analysis of slab with uniform twisting moment. (a) Slab loading and collapse mechanism, (b) Yield criteria.

3.5 Comparison with tests

Table 1 gives the ultimate twisting moment, M_{xytest}, achieved in tests on square slabs, loaded as shown in Fig 5(a) to give an applied uniform twisting moment, carried out by Marti et al.[11]. The slabs were reinforced orthogonally, with the same percentage of reinforcement, ρ_x, in the top and bottom of the slab in the x-direction and the same percentage of reinforcement, ρ_y, in the top and bottom of the slab in the y-direction, ρ_x and ρ_y could vary in the two directions. The table gives the calculated ultimate uniaxial moments, M_{xu} and M_{yu}, and the predicted moment, M_{xynorm}, using the normal moment criterion, eqn. 2. It can be seen from the ratio M_{xynorm}/M_{xytest} that the normal moment criterion is generally unsafe and that it becomes even less safe as the reinforcement ratio, ρ, increases. Some of the failures occurred due to the corner of the slab failing but the loads were considered to be close to the ultimate. Furthermore the failure of two of the slabs was due to concrete crushing i.e. over-reinforced failures; this is discussed further in the next section.

Table 1 Comparison of Marti's tests and predictions.

Slab No	ρ_x,	ρ_y,	M_{xu} kNm/m	M_{yu} kNm/m	M_{xytest} kNm/m	M_{xynorm} kNm/m	M_{xynorm} /M_{xytest}	M_{xyanal} kNm/m	M_{xytest} /M_{xyana}	M_{xynorm} /M_{xyanal}	$M_{xymorley}$ kNm/m	$M_{xymorley}$ /M_{xytest}
1	0.25	0.25	49.0	46.8	44.4	47.9	1.08	48.4	0.92	0.99	47.9	1.08
2	0.5	0.5	97.2	91.1	69.5	94.1	1.35	73.2	0.95	1.28	72.9	1.05
3	1.0	1.0	155.5	141.1	93.8*	148.1	1.58	85.6	1.09	1.73	84.4*	0.90
4	0.5	0.25	98.0	46.8	50.8	67.7	1.33	61.7	0.82	1.10	61.9	1.22
5	1.0	0.25	162.1	45.2	60.6	85.6	1.41	59.2	1.02	1.45	63.2	1.04
6	1.0	0.5	155.2	85.0	63.6	114.8	1.80	56.3	1.13	2.04	50.6*	0.80
7	0.25	0.25	43.4	40.8	42.5	42.1	0.99	42.8	0.99	0.98	42.2	0.99
8	1.0	0.25	143.1	39.6	64.8	75.3	1.16	64.1	1.01	1.17	67.5	1.04
9	1.0	1.0	142.3	129.1	101.5*	135.5	1.33	99.4	1.02	1.36	96.8	0.95

*- over-reinforced failure

4 NUMERICALLY PREDICTED YIELD SURFACES

In order to obtain yield surfaces for reinforced concrete a computer program was used. The program is capable of producing an $[M_x, M_y, M_{xy}, P_x, P_y, P_{xy}]$, $[\phi_x, \phi_y\ \phi_{xy}, \varepsilon_x, \varepsilon_y, \varepsilon_{xy}]$ response for a section up to and beyond peak load, where all forces and moments are per unit length. Full details of the analysis are given elsewhere [18]. The program has non-linear models for concrete which include non-linear stress-strain curves, biaxial stress-strain response, cracking, tension stiffening and, of particular importance, compression softening as described above. The non-linear stress-strain response of reinforcement is also modelled. In the studies to be described an elastic-plastic response was assumed. The program has been extensively tested and compared against available experimental data [18]. Figures 6 and 7 show typical results from tests carried out to validate the program.

The test results shown in Fig. 6 were obtained by May et al [13] from a slab element 1.6m square and 150mm deep with approximately 0.95% of reinforcement in the top and 0.95% in the bottom, based on the total depth, placed in the direction of the applied moment. The slab was loaded with bi-axial moments M_x and M_y of equal magnitude but opposite sign. The figure shows a comparison of the experimental and analytically predicted applied moment against average curvature. The test results shown in Fig. 7 were obtained by Marti et al [11] from a slab loaded in pure torsion, M_{xy}. The slab was 1.7m square 200mm deep with 1% of reinforcement in each of the orthogonal directions in the top and 1% of reinforcement in each of the orthogonal directions in the bottom. The figure shows a comparison of the experimental and analytically predicted applied torsional moment against average torsional curvature. The results show excellent agreement but also indicate how when torsional moments are applied the mode of failure, for even comparatively low proportions of reinforcement, can be non-ductile. Further discussion of the modes of failure is given later. A summary of the ultimate loads, M_{xyanal}, obtained using the computer program is given in Table 1 together with a comparison with the test results, which shows reasonable agreement. Tests 3 and 9 were over reinforced failures. This would not have occurred for slabs with 1% reinforcement in each face and both directions had the loading resulted in bending predominantly about the reinforcement directions, see for example Fig 6.

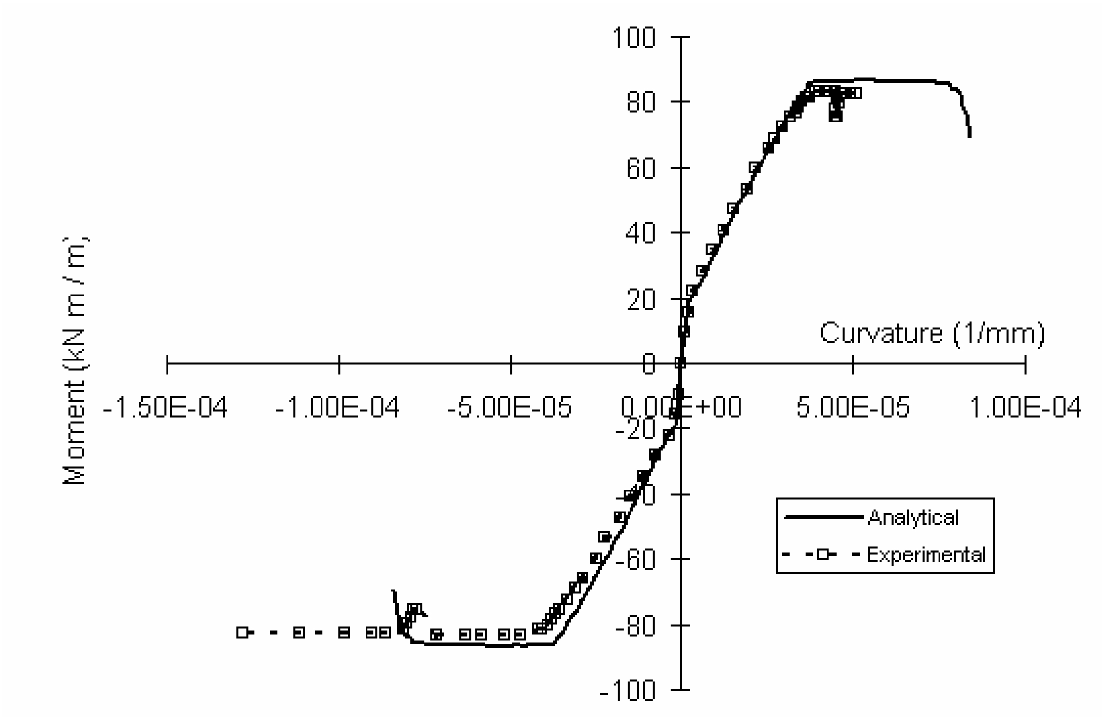


Fig. 6 Applied moment curvature relationship for biaxially loaded slab element. [12]

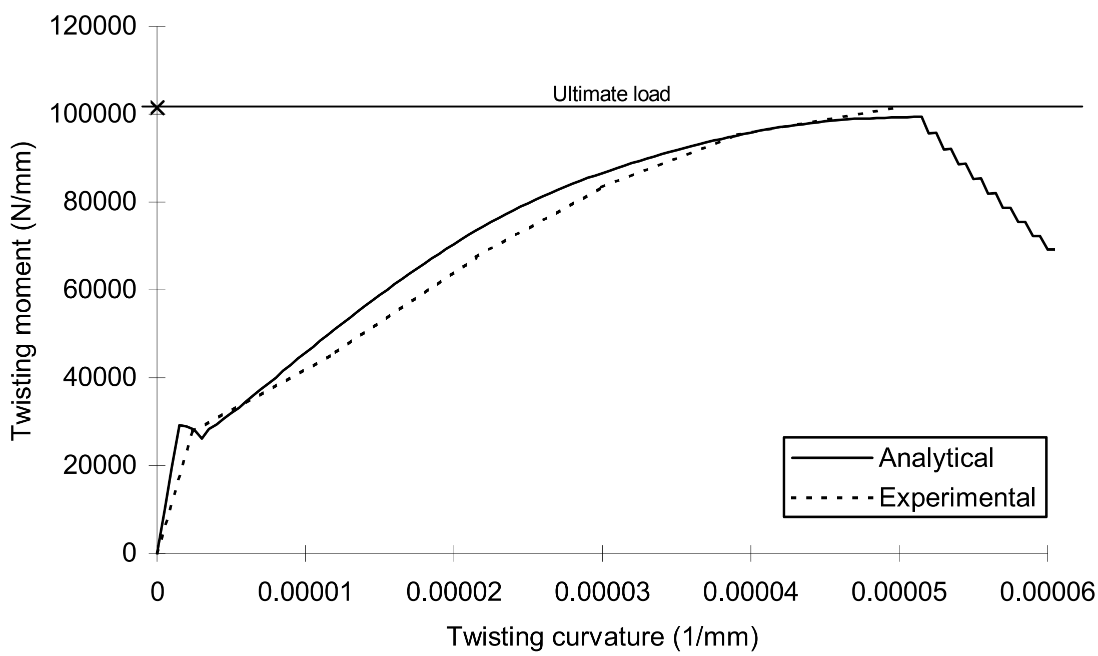


Fig. 7 Applied twisting moment curvature relationship for slab [10].

5 GENERATION OF YIELD SURFACES

To generate the yield surfaces the in-plane loads P_x, P_y and P_{xy} have been set to zero and specified ratios of M_x, M_y and M_{xy} given; these have been incremented up to and beyond their maximum values. The maximum values have then been used to produce surfaces similar to that in Fig 8, where values have been divided by the uniaxial ultimate moment, M_{uni}. A number of surfaces have been produced for differing reinforcement values for isotropic and orthotropic doubly reinforced sections and with section and material properties used in the analysis are given elsewhere [18].

5.1 Isotropic reinforcement

Yield surfaces for a section with 0.5% and 1.0% reinforcement in each of the orthogonal directions in both the top and bottom of the section are shown, Figs 8 and 9. The surfaces are plotted in terms of principal stresses with θ being the orientation of the principal planes with respect to the x-axis. Also shown are the predictions of the normal yield criterion, which in this case, since the section is isotropically reinforced, is the square yield criterion.

For 0.5% reinforcement it can be seen that the yield criterion is satisfactory with the largest errors being for $M_1 = - M_2$ and $\theta = 45°$, i.e. pure twisting, when the yield criterion over-estimates the yield moments by about 10%. When $M_1 = M_2$ the yield criterion is slightly conservative, due to the increased compressive strength when the concrete is in biaxial compression.

For 1.0% reinforcement, Fig 9, the yield criterion is satisfactory if the principal moments are both of the same sign. However, if the moments are of opposite sign the yield criterion is increasingly more un-conservative as θ increases. When $\theta = 45°$, i.e. pure twisting, the yield criterion over-estimates the twisting moment by a factor of 2.

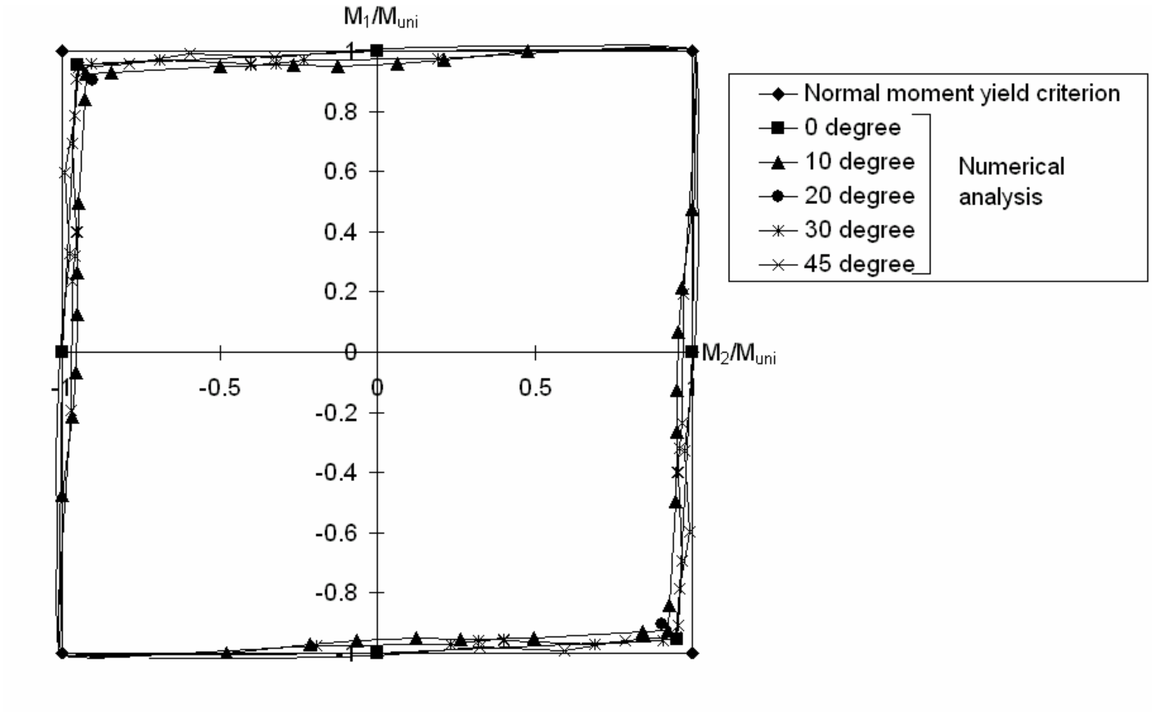


Fig. 8. Failure surfaces for 0.5% reinforcement in each direction top and bottom.

Because the numerical analyses can follow the response of the section beyond the peak load it is possible to investigate the ductility of the section under the various types of loading. Fig 10 shows the response of the section with 1% reinforcement for a principal moment ratio of –1. The most important observation to be made is that as θ changes from 0 to 45° the mode of failure changes from an under-reinforced failure to an over-reinforced failure with consequent loss of ductility. This implies that plasticity theory would not be valid for such elements. As far as the authors are aware this observation has not been shown experimentally because of the difficulty of following the falling branch of the load deflection response, although Marti et al [11] did report over-reinforced failure in some of their tests, Table 1. This has been also noted by May et al [13].

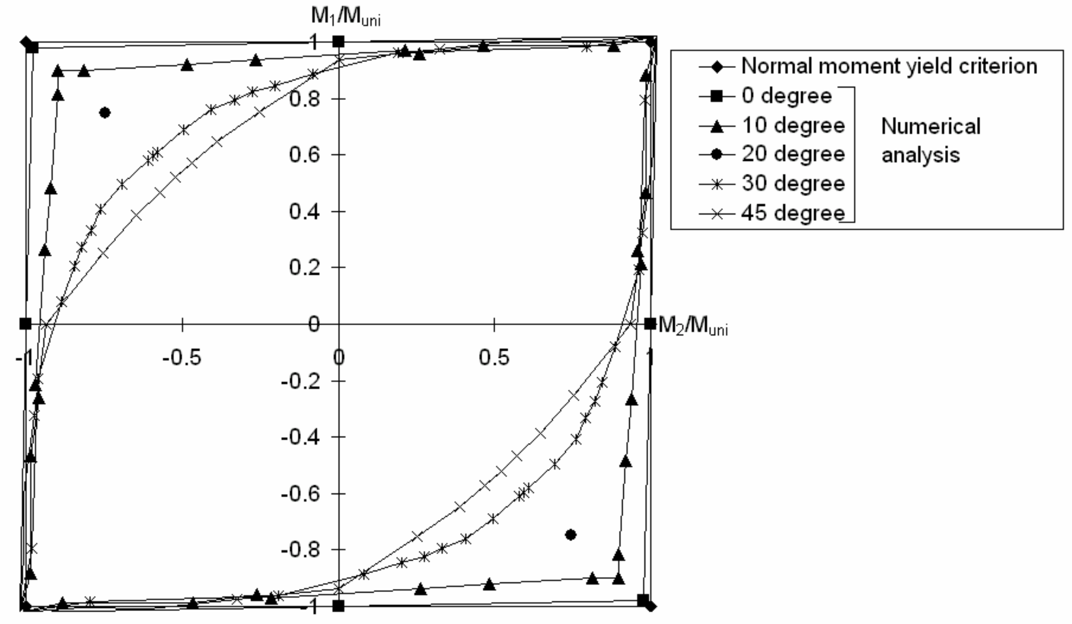


Fig. 9. Failure surfaces for 1% reinforcement in each direction top and bottom.

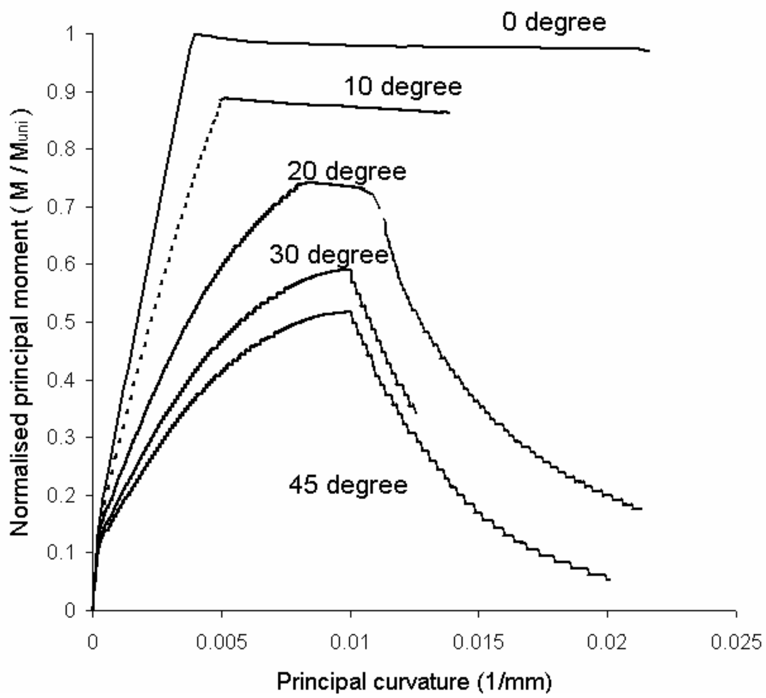


Fig. 10. Moment curvature relationships for 1% reinforcement in each direction top and bottom. $M_1/M_2 = -1$

6 PRACTICAL CONSIDERATIONS

This section attempts to identify how the behaviour of slabs is affected by the shortcomings in the yield criterion. In tests of many slabs the general conclusion has been that the use of yield line theory, which uses the normal moment yield criterion, although an upper bound approach produces estimates of loads that are conservative. This has been confirmed by finite element studies, e.g. May and Ganaba [19]. The reasons for this conservatism are that in general the areas of reinforcement used are low and that internally equilibrated membrane forces exist. Normally yield line analysis uses mechanisms that ignore membrane effects; strictly speaking most mechanisms are not kinematically admissible for a section in which the neutral axis is not at the centre of the section. Yield line analyses that include the membrane effects predict the increase in load capacity [20].

However, it is possible to have slabs in which membrane forces are not generated and which can have substantial twisting moments and designers need to be aware of the un-conservatism of the normal moment yield criterion.

It is of some concern that the Wood-Armer equations, which have generally been considered to be a lower bound and therefore safe approach, suffer from the same problems as above in that slabs with reasonably low areas of reinforcement, approximately 1%, could have their strength underestimated if they have significant areas under predominantly twisting moments and cannot generate membrane action.

The Morley and Gulvanessian sandwich approach combined with the Clark-Nielsen equations is able to model successfully the twisting moment capacity and the uni-axial moments and is thus is able to model a yield criterion close to that determined numerically. It also has the advantage of indicating more accurately the depth of the neutral axis and hence gives an indication of when the failure is likely to be over-reinforced and hence lack ductility. The author would therefore propose that in general this approach should be used rather than the Wood-Armer equations. If the latter are being used then users need to be aware of the shortcomings of such an approach when the principal moments are of opposite sign.

7 CONCLUSIONS

Some shortcomings have been identified in the normal moment yield criterion for reinforced concrete slabs. For the majority of slabs the un-conservatism in the yield criterion is outweighed by the effect of membrane action, which is generally ignored. However, in order to avoid problems with the yield criterion and possible lack of ductility, the Morley and Gulvanessian sandwich approach combined with the Clark-Nielsen equations is considered to be the best procedure. In response to the question posed in the title the author can see no reason why Morley's sandwich should not be used and a number of reasons, identified above, why it should be.

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APPENDIX 1

Morley and Gulvanessian's sandwich approach

This approach is normally applied to shell elements with the six stress resultants, moments M_x, M_y, M_{xy}, and in-plane forces P_x, P_y, P_{xy}. The forces are replaced by two sets of statically equivalent in-plane forces, N_x, N_y, N_{xy}, acting on two layers one at the top and one at the bottom of the element. The tests carried out by Marti were on elements under pure twisting thus M_x, M_y, and the in-plane forces P_x, P_y, P_{xy} were all zero, therefore M_{xy} can be replaced by N_{xy} and $-N_{xy}$ acting on the top and bottom layer of the sandwich shell. The Clark-Nielsen Equations, see below, can now be used to determine the ultimate values of N_{xy}. In order to determine M_{xy} the lever arm of the in-plane shears is taken as the distance between the centroids of the concrete stress blocks.

Clark-Nielsen equations

In order to derive the equations it is assumed that there is a plane along which failure occurs due to yielding of the reinforcement in tension, Fig A1.

The ultimate strengths in the x and y directions respectively are given by

$$N_x^* = A_x f_s \tag{A1}$$
$$N_y^* = A_y f_s \tag{A2}$$

Then the strength in the n direction is:

$$N_n^* = N_x^* \cos^2\theta + N_y^* \sin^2\theta \tag{A3}$$

The applied force in the n direction is:

$$N_n = N_x \cos^2\theta + N_y \sin^2\theta - 2N_{xy} \sin\theta\cos\theta \tag{A4}$$

It can be shown that for the general case in which tension reinforcement is required in both directions

$$N_x^* = N_x - kN_{xy} \tag{A5}$$

$$N_y^* = N_y - \frac{1}{k}N_{xy} \tag{A6}$$

where *k* is tanθ.

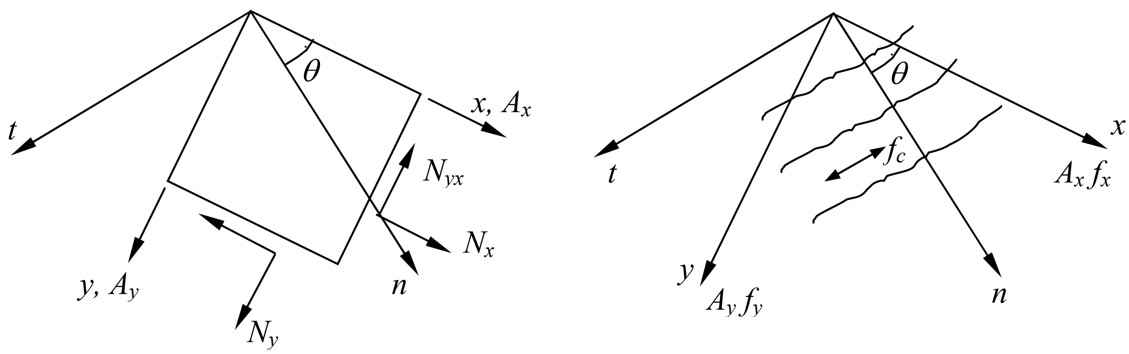

Fig A1 Clark Nielson equations. (a) Axes and in-plane forces. (b) Failure direction.

Normally for design equations A5 and A6 are optimised to give the minimum area of reinforcement in which case the equations become

$$N_x^* = N_x + |N_{xy}| \tag{A7}$$

$$N_y^* = N_y + |N_{xy}| \tag{A8}$$

However, the equations can be recast in order to obtain the strength for a given set of reinforcement. In order to analyse the panels tested by Marti since they were loaded in pure torsion then N_x and N_y are both zero. Thus from equations A5 and A6

$$k = \tan\theta = \sqrt{\frac{N_x^*}{N_y^*}} \tag{A9}$$

On the t plane equilibrium gives

$$N_t^* = N_x^* \sin^2\theta + N_y^* \cos^2\theta + F_c \tag{A10}$$

where F_c is the force per unit length in the concrete.

The applied force is:

$$N_t = N_x \sin^2\theta + N_y \cos^2\theta + 2N_{xy}\sin\theta\cos^2\theta \tag{A11}$$

Since $N_x = N_y = 0$

$$F_c = -N_x^* \sin^2\theta - N_y^* \cos^2 - 2N_x^* \sin^2\theta \tag{A12}$$

Hence the depth of the stress block, e, is given by

$$e = F_c/\sigma_c \tag{A13}$$

where σ_c is the ultimate compressive stress in the concrete.

Solution procedure

The solution procedure adopted for the analysis was as follows

N_x^* and N_y^* were calculated from equations A1 and A2
θ was determined from equation A10
The concrete force was determined from equation A12
The depth of the compression block was determined using equation A13.
Hence N_{xy} can now be determined from equation A5 or A6 with N_x or N_y equal to zero

M_{xy} is then given by:

$$M_{xy} = N_{xy}(h-e) \qquad \text{(A14)}$$

If the depth of the stress block is greater than half the depth of the section then N_{xy} is reduced in the ratio h/2e to give N_{xyr} , the maximum concrete shear stress and M_{xy} is given by

$$M_{xy} = N_{xyr} h/2 \qquad \text{(A15)}$$

This would also imply an over- reinforced failure.

APPENDIX 2

WOOD-ARMER EQUATIONS

The Wood-Armer equations used for the design of reinforcement in slabs in bending are derived using the normal moment yield criterion and can be written as

$$M_x^{*t} = M_x - k_t M_{xy}$$

$$M_y^{*t} = M_y - k_t M_{xy}$$

$$M_x^{*b} = M_x + k_b M_{xy}$$

$$M_y^{*t} = M_y + k_b M_{xy}$$

In the case of pure twisting

$$M_x = M_y = 0$$

For the cases tested by Marti

$$M_x^{*t} = -M_x^{*b} \text{ and } M_y^{*t} = -M_y^{*b}$$

Hence

$$M_{xy} = -M_x^{*t}/k \quad k = -M_{xy}/M_y^{*t}$$

Thus

$$M_{xy} = \sqrt{\left(M_x^{*t} M_y^{*b}\right)}$$