THE PLASTICITY OF UNREINFORCED CONCRETE

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1 THE MATERIAL

If a material is to be structurally useful it should be ductile. That is, steel, under-reinforced concrete, aluminium alloys and perhaps wrought iron are acceptable, but cast iron and glass are not – they would shatter if incorporated as load-bearing members in a practical structure. Concrete is brittle, and it is the steel (or other) reinforcement that confers ductility on reinforced-concrete beams, columns and slabs. It is not obvious that concrete without reinforcement can be used safely in construction.

However, such a view is contradicted immediately by the continued existence, for example, of the Roman Pantheon, after nearly two millennia. At 43 m the mass concrete dome spans a metre more than the stone masonry domes of St Peter's, Rome and of Florence. Like those domes it has cracked in response to small movements of the supporting substructure; like those domes, despite the cracking, it is perfectly comfortable under the action of its own weight and of superimposed load – wind and snow. It is indeed the cracking that has conferred ductility on the unreinforced concrete (and on the stone masonry); the domes have been able to accommodate (unforeseen and unknown) changes in their "boundary conditions".

Cracked unreinforced concrete behaves, in fact, like cracked stone masonry, and its behaviour may be interpreted in the light of the plastic theorems. These theorems, developed initially for steel structures and applied later to reinforced concrete, are valid if used carefully for any ductile structural material.

2 THE MASONRY ARCH

A very simple masonry structure is the single-span arch, which could in theory, and in fact in practice, be cast from mass concrete. However, attention will be confined for the moment to more traditional construction, and fig. 1(a) shows a schematic but reasonably realistic masonry arch formed from wedge-shaped stone voussoirs. The mortar between the voussoirs is usually very weak in tension, and may in practice be absent, so that, in reality there is nothing to prevent the stones pulling apart. By contrast, the compressive forces between the voussoirs are so small that resulting stresses are very low – even a large-span bridge will experience stresses well under 10 per cent of the crushing strength of the stone. With these remarks in mind, the material properties of masonry may be formulated, and there are three key assumptions, stated for the voussoir arch, but applicable to other structural forms (for example, the elements – towers, spires, vaults, buttresses and so on – of a great church):

2.1 Sliding failure cannot occur

It is assumed that friction is high enough between voussoirs, or that the stones are otherwise effectively interlocked, so that they cannot slide one on another. It turns out that this is a reasonable assumption, although it is certainly possible to find occasional evidence of slippage in a masonry structure.

2.2 Masonry has no tensile strength

Stone itself has a definite tensile strength, but it is the joints between the stones that are weak. Thus the assumption implies that only compressive forces can be transmitted between masonry elements. In accordance with common sense, and with the principles of the plastic theorems (discussed below), this assumption is "safe".

2.3 Masonry has an infinite compressive strength

This assumption is a consequence of the fact that, in practice, stresses are far removed from the crushing strength of the material. The assumption is obviously “unsafe”, but it is not unrealistic; it is discussed further below.

Thus a picture emerges of masonry as an assemblage of stones shaped to pack together into a coherent structural form, with that form maintained by compressive forces transmitted within the mass of the material. (Since stresses are low, the term “masonry” includes not only stone and weaker bricks, but also say breeze blocks and more primitive materials such as sun-dried mud.) The question then arises as to how such a masonry assemblage might fail in any meaningful structural sense. If the masonry is infinitely strong, then it would seem that a calculation of levels of compressive stress will not be relevant. The idea that tension is not permissible is, however, significant.

The arch of fig. 1(a) would be constructed on temporary falsework or “centering”; when the keystone has been placed, the centering may be removed, and the arch immediately starts to thrust against its abutments, and those abutments (the river banks, say) will inevitably give way. The arch, composed of strong, virtually rigid, voussoirs, must adapt to a slightly increased span, and it does this by cracking at the joints, shown greatly exaggerated as “hinges” in fig. 1(b). Thus the arch is freely deformable to conform to the new span, and, despite the brittleness of the individual elements, the structure as a whole exhibits “ductile” behaviour.

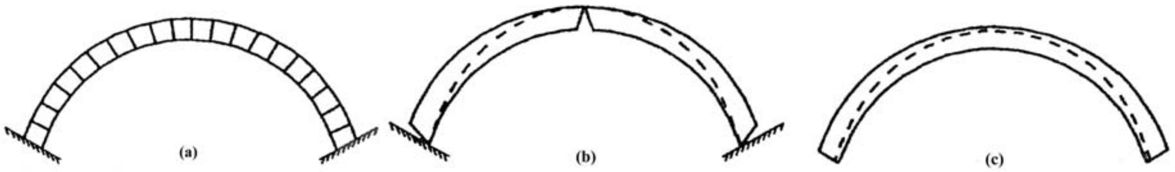

Fig. 1 Masonry voussoir arch: (a) as built, (b) after spread of the abutments, (c) general position of line of thrust.

The compressive structural forces must of necessity pass through the hinge points of fig. 1(b), and the broken line in the figure represents what may be called loosely the line of thrust, that is, the resultant of the compressive forces passed from voussoir to voussoir within the masonry profile. If a particular joint *Mm* is examined, fig. 2, then the structural action at the joint will be specified in terms of the magnitude, direction and point of application of the force transmitted across the joint. The tangential component of the force is not of importance, since slip is assumed not to occur – what is needed is the value N of the normal force across the joint together with the value of its eccentricity e from the centre line. It is convenient to work with a “bending moment” $M = Ne$ as a second variable, so that the stress resultants M, N define the state of the arch at any particular section.

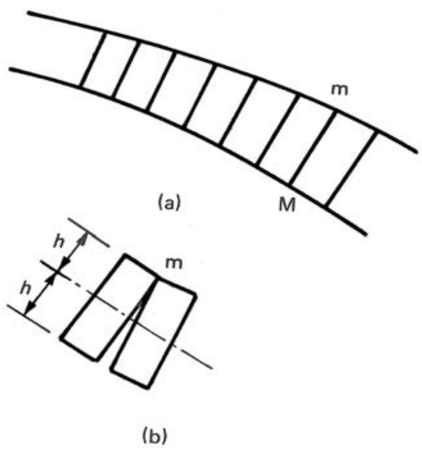

Fig. 2 A hinge forming in a voussoir arch.

The hinge of fig. 2(b) will form when the eccentricity e of the normal thrust just has the value h, that is when $M = hN$. The lines $M = \pm hN$ are shown as *OA* and *OB* in fig. 3(a), and they represent, for any given joint between voussoirs, the condition that a hinge is in existence at that joint. A general point (N, M) in the figure which lies within the open triangle *AOB* represents a thrust between voussoirs at an eccentricity less than h, that is, the line of thrust lies within the voussoirs at that joint and no hinge is forming. If the general point lies on *OA* or *OB*, then a hinge is forming in either the intrados or extrados of the arch. The general point cannot lie outside the region *AOB*, since this would imply tension at the joints.

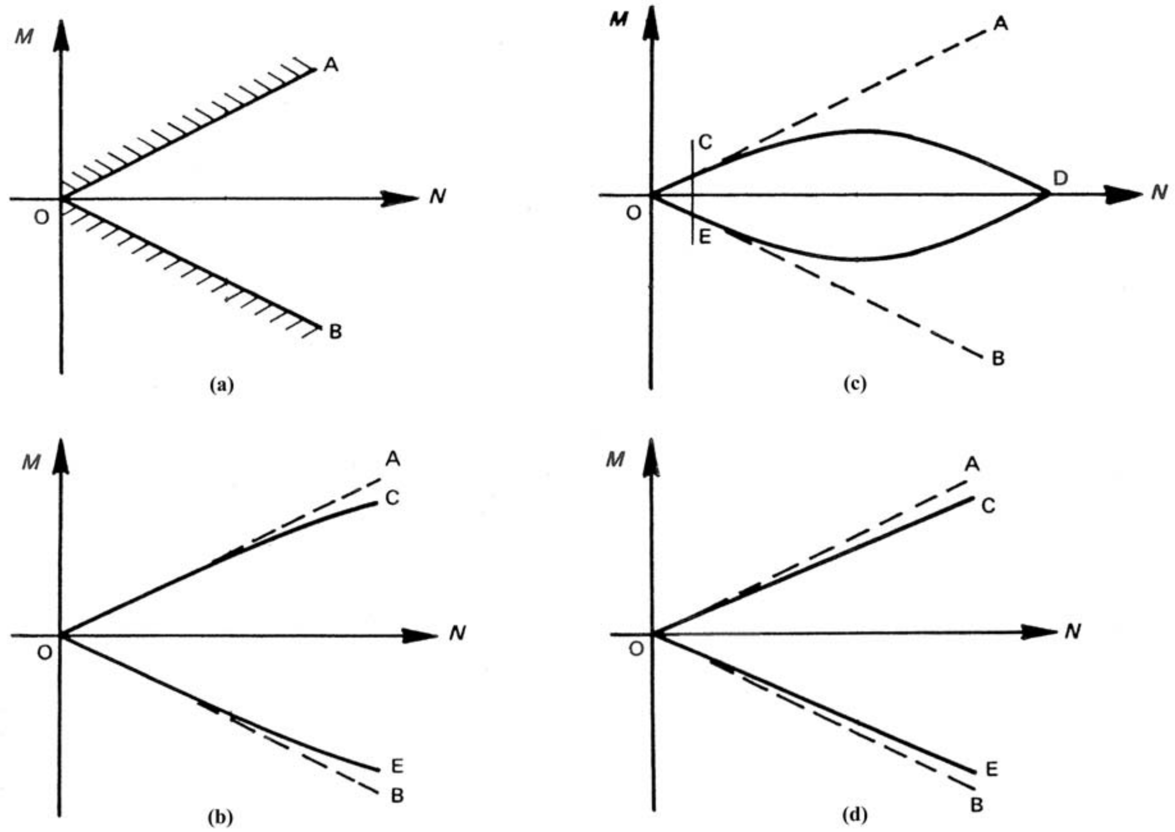

Fig. 3 Yield surfaces for masonry. (a) Material infinitely strong. (b) and (c) Material with finite crushing strength. (d) Approximate yield surface.

The construction of fig. 3(a) involves the assumption that the material has infinite compressive stress. As the line of thrust approaches the edge of a voussoir so the stress on the diminishing area of contact will increase, and a real stone with a finite crushing strength will not permit the line of contact at a hinge that is illustrated in fig. 2(b). Thus the lines *OA* and *OB* in fig. 3(a) cannot quite be reached, they are replaced by the slightly curved lines of fig. 3(b). The full boundary is formed by the parabolic arcs *OCD* and *OED* in fig. 3(c), and the general point (*N, M*) must lie within this boundary. The assumption of low mean compressive stress in fact constrains the point (*N, M*) to lie within an area such as *OCE* in fig. 3(c), and it is this area which is enlarged in fig. 3(b).

The sketches shown in fig. 3 are of course yield surfaces of plastic theory, and plastic principles may be applied. A general point (*N, M*) lying within the full yield surface of fig. 3(c) represents a safe state for the masonry. The curved boundaries of fig. 3(b) also represent a “safe” yield surface, and they may be straightened by the device shown in fig. 3(d). If the mean compressive stresses are known not to exceed 10 per cent of the crushing strength of the stone, then the straight lines *OA* and *OB* may be replaced by $M = \pm 0.9hN$. Thus the real arch having a (local) ring depth of $2h$ is replaced, for the purposes of analysis, by a hypothetical arch of depth $2(0.9h)$. (This kind of “shrinking” is of some importance in assessing the safety of masonry arches.)

The abutments of the arch in fig. 1 were imagined to move apart, leading to the pattern of cracks sketched in fig. 1(b). The originally hyperstatic structure (with three redundancies) has been transformed into a “three-pin” arch, which is now statically determinate; for the known loading, the value of the abutment thrust may be determined (and is, in fact, the lowest value which will maintain the arch in equilibrium). The three-pin arch is a well known satisfactory structural form – the development of three hinges by cracking of the joints does not presage collapse.

If the abutments of the arch do not move apart, or move apart only slightly, and the voussoirs are almost but not absolutely rigid, then the joints between voussoirs will remain tight, and no hinges will occur to help locate the line of thrust. All the designer may be able to show is that the line of thrust occupies some such position as that sketched in fig. 1(c), where it lies completely within the masonry. In fact, this is all the designer needs to show. If any one position such as that shown in fig. 1(c) can be found, then this is absolute proof, by the "safe theorem" of plastic theory, that the structure is safe. If the designer can determine a way in which the structure can carry the given loads, then the structure can certainly also find a way.

This anthropomorphic statement does not, of itself, give any indication of how safe the structure might be. Since the masonry has been assumed to be of infinite compressive strength, there is no question of failure of the material. Instead, a geometrical criterion can be devised. As sketched in fig. 1(c), the shape of the line of thrust is not the same as the shape of the profile of the arch, and there is a minimum thickness of the arch which will only just contain the line of thrust.

Robert Hooke in 1675 identified the shape of the line of thrust by his statement: As hangs the flexible line, so but inverted will stand the rigid arch. In other words, if the given loading for the arch were applied to a light string, then the shape of that string, in tension, would be the same, inverted, as that of the arch to carry the same loads in compression. In fig. 1(c), for example, if the loading resulting from the circular profile of the arch were uniform, then the line of thrust sketched in the figure would be the mathematical catenary. An arch built with a thickness just able to contain the catenary would be precariously stable, whereas an arch of double that thickness would easily accommodate a wide range of possible lines of thrust. In practice, a "geometrical factor of safety" of 2 appears to be appropriate, to allow for building irregularities and for movements imposed by the environment, both of which can distort the original designed geometry.

3 UNREINFORCED CONCRETE

Account may be taken of the fact that mass concrete has some small (rather than zero) tensile strength, in order to embrace its behaviour more formally within the framework of plastic theory. The arch of fig. 1 may be imagined to be cast from unreinforced concrete; spread of the abutments, fig. 1(b), will impose tensile strains on the material. Where those strains are greatest cracks will occur, and the "hinge" pattern of fig. 1(b) will be developed. The small tensile strength of the material, will, however, ensure that the concrete elsewhere in the arch remains coherent and, instead of a monolith, the arch is now composed of two members. The statical arguments developed for the stone voussoir arch are unchanged, and the broken line in fig. 1(b) represents the line of action of compressive forces passed within the thickness of the (two-member) arch ring.

The assumption of a small tensile strength for the material is necessary for the integrity of the structure. Without such an assumption, the concrete would disintegrate into small particles – the arch of fig. 1 cannot be constructed in dry sand. Provided, however, that the mass concrete structure remains a structure, cracked or not, then its analysis may be undertaken by the use of plastic theory.

Above all, it is the "safe theorem" that is the basic tool for the engineer. The presentation given above has been for the simple arch, but any mass concrete structure, however complex, is safe if it can contain compressive forces in equilibrium with the external loading. All the designer needs to show is that at least one such system is possible.

4 THE ROMAN PANTHEON

Masonry domes are, almost without exception, cracked (the exceptions seem to be Wren's double masonry domes of St Paul's Cathedral). The cracks are sketched as a regular pattern in fig. 4, and the actual cracks at St Peter's, Rome, and the octagonal dome of Santa Maria del Fiore, Florence, conform closely to this configuration. The cracks are caused by the spread of the supporting cylindrical drum (cf. the cracking of the two-dimensional arch due to the spread of the abutments fig. 1), and follow meridional lines, running out as they approach the crown. Indeed, the crown remains solid, and is in effect supported by a (large) number of tapering quasi two-dimensional (half) arches. The inner dome of St Paul's has an "eye" in its crown, and the tapering arches effectively support a horizontal ring of masonry subjected to compressive hoop stresses.

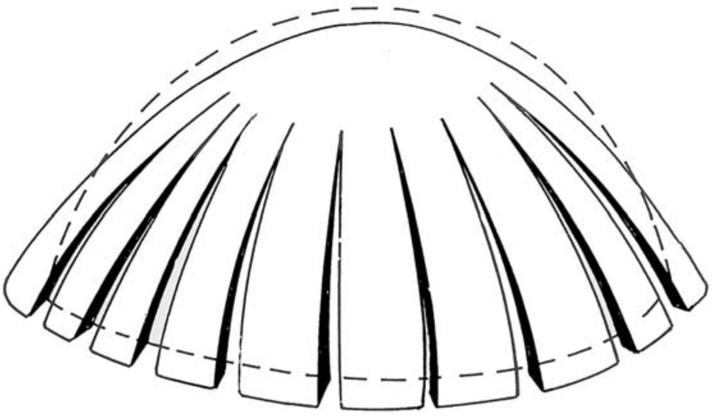

Fig. 4 Greatly exaggerated schematic illustration of cracking of a dome due to increase of span.

The Roman Pantheon (c. AD 120) also has an eye, and is cracked in the same meridional way; this dome, however, was constructed from mass concrete. The cross-section shown in fig. 5 must be interpreted as representing a series of tapering arches – “orange slices” – composed of mass concrete. Each of these slices may be analysed as a two-dimensional arch, and it is a simple matter to construct lines of thrust lying wholly within the masonry, and thus to confirm that it is possible to calculate sets of forces for the concrete which are purely compressive. Moreover, the analysis will determine close estimates of the horizontal thrust that the dome exerts on its supporting cylindrical structure.

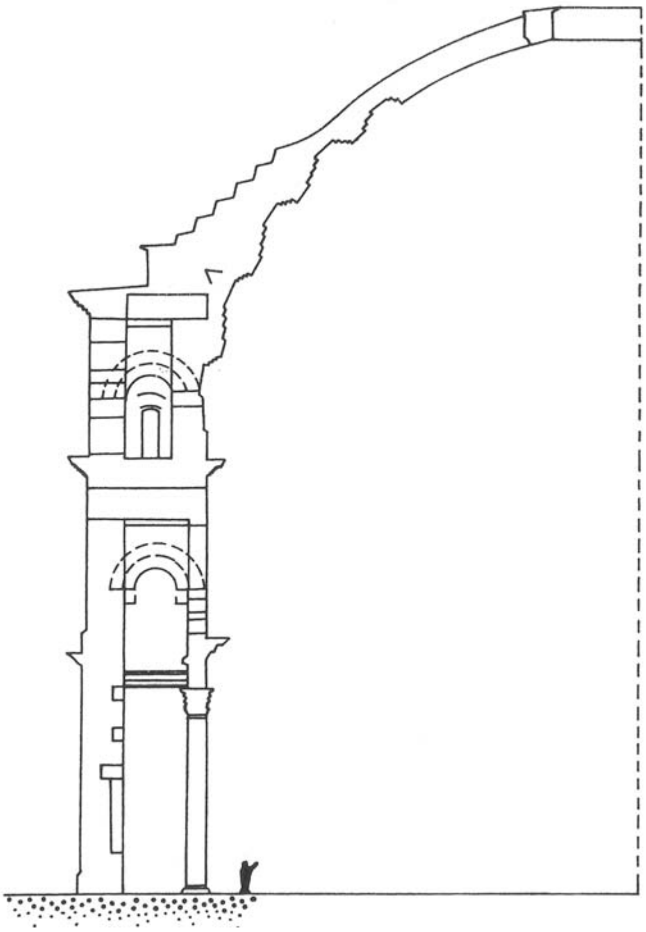

Fig. 5 Cross-section of the Pantheon, Rome (AD 120).

5 CONCLUSIONS

- Engineers use a range of materials and many forms in the design of structures. All such structures have the common property of ductility – they can deform, within acceptable limits, without any drop in the applied loading. Ductility is conferred by the "plasticity" of the materials (e.g. mild steel, reinforced concrete, aluminium alloy) or by the behaviour of the structure itself (premature collapse by unstable buckling must be prevented).
- Mass concrete, with small resistance to tensile stress, will crack when used in a practical structure. Such cracking, far from presaging collapse, divides the structure into "masonry" blocks, and the structure as a whole can deform in a ductile way to accommodate imposed movements from the environment.
- Structures made from mass concrete may be analysed by the use of plastic principles as applied to stone masonry. Above all, if *any* set of forces can be found with which the structure is "comfortable", then this is assurance that the structure is safe.