APPLICATION OF PLASTICITY THEORY TO REINFORCED CONCRETE DEEP BEAMS

Ashraf ASHOUR [1] Keun-Hyeok YANG [2]

[1] EDT1, School of Engineering, Design and Technology, University of Bradford, UK.
[2] Department of Architectural Engineering, Mokpo National University, South Korea.

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1 INTRODUCTION

Reinforced concrete deep beams are fairly common structural elements. They are characterised as being relatively short and deep, having a small thickness relative to their span or depth. Typical applications of deep beams include transfer girders, pile caps, tanks, folded plates and foundation walls, often receiving many small loads in their own plane and transferring them to a small number of reaction points.

Elastic solutions of reinforced concrete deep beams provide a good description of the behaviour before cracking, but after cracking, a major redistribution of stresses occurs and hence the beam capacity must be predicted by inelastic analysis. Schlaich et al. [1, 2] identified deep beams as discontinuity regions where the strain distribution is significantly nonlinear and specific strut-and-tie models need to be developed, whereas shallow beams are characterised by linear strain distribution and most of the applied load is transferred through a fairly uniform diagonal compression field.

This paper reviews the application of the plasticity theory to reinforced concrete deep beams. Both lower and upper bound theorems of the plasticity theory were employed to predict the capacity of reinforced concrete deep beams. Most current codes of practice [3-6], for example ACI 318-05 and Eurocode 1992, adopt strut-and-tie approach for designing reinforced concrete deep beams.

2 PLASTICITY THEORY FOR REINFORCED CONCRETE

The plasticity theory for rigid plastic structures mainly comprises the lower-bound theorem, the upper-bound theorem, and the uniqueness theorem [7, 8]. To apply the lower-bound theorem, it suffices to find a load path to transfer the load to support, satisfying the yield criteria throughout the structure, and then equilibrium equations are applied to calculate the capacity. By contrast, in the upper-bound analysis, a kinematically admissible failure mechanism would be identified and then the energy principle is used to provide an upper-bound load on the capacity. The uniqueness theorem states that the lowest upper-bound solution and the highest lower-bound solution coincide and constitute the exact collapse load of the structure.

The application of the theory of plasticity to under-reinforced concrete structures, where the failure is governed primarily by yielding of steel reinforcement, seems reasonable and such application has a fairly long history. The yield line theory developed by Johansen and Gvozdev (quoted from [7, 8]) gives good prediction of the failure load of reinforced concrete slabs. Also, the prediction of flexural capacity of reinforced concrete beams and frames using limit analysis is well established in most codes of practice. However, it is far less obvious that the theory of plasticity can be applied to over-reinforced or unreinforced concrete structures where the behaviour is governed mainly by concrete as a typical stress-strain curve of a plain concrete specimen under compression is characterised by strain softening and not by the yield plateau that would normally be required to apply the theory of plasticity. Nevertheless, remarkably good correlation between plasticity analysis of such reinforced concrete structures and test results has been achieved when the concrete strength is modified by a reduction factor called the effectiveness factor [7-19].

The capacity of reinforced concrete deep beams is governed mainly by shear strength owing to their geometric proportions. The problem of shear in reinforced concrete was extensively studied for about a century to develop a rational procedure to predict the shear strength of reinforced concrete beams [20]. It was suggested [21] that the extension of plasticity theory would be one of the most important advances in understanding shear in reinforced concrete. In particular, the mechanism analysis of shear failure that considers translations, rotations, and combinations of the two would give a clear picture of shear in reinforced concrete structures [20].

The plastic theory was first applied to shear in reinforced concrete structures by research groups in Denmark [8, 22] and Zurich [23]. Design philosophies for shear in reinforced concrete beams using plasticity theory comprise two branches; the truss analogy and the kinematic approach. The kinematic approach is far less developed than the static approach. In the following, the advances in the mechanism and strut-and-tie analyses for reinforced concrete deep beams are reported and discussed.

3 MODELLING OF MATERIALS

Narrow reinforced concrete deep beams can reasonably be assumed to be in a state of plane stress. Concrete is modelled as a rigid-perfectly plastic material obeying modified Coulomb failure criteria [7, 8], as shown in Fig. 1 (in the principal axes). The effective compressive strength f_c of concrete to be used in calculation can be obtained from the cylinder compressive strength $f_c^{'}$ by introducing the effectiveness factor ν as given below:

$$f_c = \nu\, f_c^{'} \tag{1}$$

Similarly, the tensile strength would also be modified by an effectiveness factor to give the plastic tensile strength. However, as ductility of concrete in tension is very limited, this effectiveness factor should be very low and, in most cases, tensile strength of concrete is ignored.

Steel reinforcement in both tension and compression is assumed as a rigid perfectly plastic material with yield strength f_y. Perfect bond between concrete and steel reinforcement has to be assumed in mechanism analysis, whereas full end anchorage of steel reinforcement should be fulfilled in strut and tie models of deep beams.

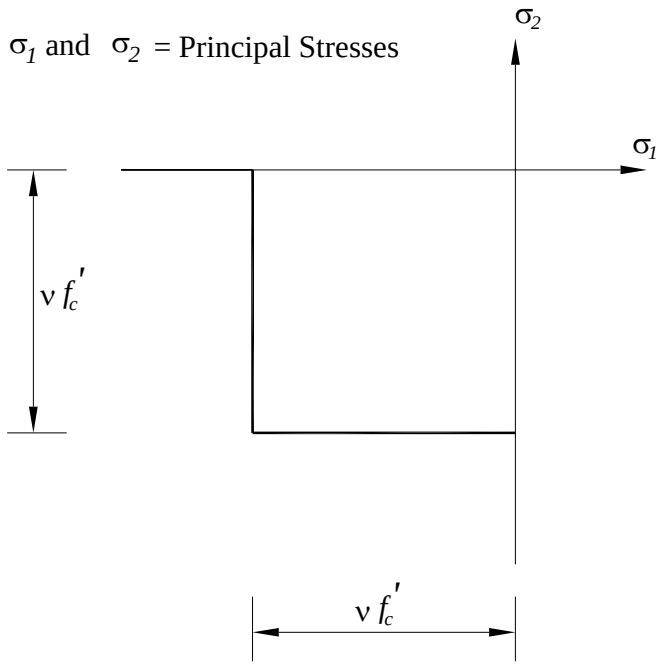


Fig. 1 Modified Coulomb Failure Criteria with zero tension cut-off for Concrete (in principal axes).

4 Effectiveness Factor of Concrete

The reason for introducing the effectiveness factor ν in Eq. (1) above is to account for the limited ductility of concrete and to absorb other shortcomings of applying the theory of plasticity to concrete. The value of the effectiveness factor of concrete normally depends on the material, size, geometry,

reinforcement and loading of the structure [8]. The application of the plasticity theory would be useful to the extent that the effectiveness factor is reasonably uniform and predictable across a range of specimens.

Different techniques were suggested for evaluating the effectiveness factor of concrete. Exner [24] proposed a computational method to calculate the effectiveness factor by comparing the area below the real uniaxial stress-strain curve for concrete in compression and an elastic-perfectly plastic stress-strain curve. The value of the effectiveness factor obtained is a function of the shape of the real stress-strain curve, compressive strength and the ultimate strain of concrete. Nielsen et al. [8, 25] proposed a formula for the effectiveness factor of concrete beams failing in shear as a function of cylinder compressive strength:

$$\nu = 0.8 - \frac{f_c^{'}}{200} \qquad (f_c^{'} \text{ in } N/mm^2) \tag{2}$$

The above formula indicates that increasing the concrete strength reduces the value of the effectiveness factor as the lower the concrete strength, the flatter the stress-strain curve, and the more observed ductility.

Although the plastic behaviour of reinforced concrete structures is mainly influenced by the amount of reinforcement, very few formulae proposed for the effectiveness factor consider the amount of reinforcement. Based on statistical analysis for the effectiveness factor of concrete in continuous deep beams, Ashour and Morley [26] proposed a formula for the effectiveness factor of concrete in continuous deep beams expressed in terms of concrete strength and amount of reinforcement:

$$\nu = 0.7 - \frac{f_c^{'}}{110} - \frac{\rho}{0.85} \qquad (f_c^{'} \text{ in } N/mm^2) \tag{3}$$

where ρ is a weighted-reinforcement ratio for the horizontal and vertical reinforcement based on their relative contribution to the load capacity of continuous deep beams.

Based on tests of concrete panels under shear, Vecchio and Collins [27] related the effectiveness factor to the tensile strain normal to the principal compressive strain as expressed below:

$$\begin{aligned} &\nu = \frac{1}{1.0 + k_c k_f} \\ &\text{where} \\ &k_c = 0.35\left(-\frac{\varepsilon_1}{\varepsilon_3} - 0.28\right)^{0.8} \geq 1.0 \\ &k_f = 0.1825\sqrt{f_c^{'}} \geq 1.0 \end{aligned} \tag{4}$$

and ε_1 and ε_3 = the principal tensile and compressive strains, respectively.

In many investigations, the value of the effectiveness factor of concrete would be simply calculated by calibrating the failure loads obtained from the plasticity analysis against those from experiments. This technique shows significant variation of the effectiveness factor for different reinforced concrete structures. According to Oesterle et al. [28], the value of the effectiveness factor for concrete wall structures failing in shear varies between 0.16 and 0.49. Rogowsky and MacGregor [29] proposed values for the effectiveness factor varying from 0.25 to 0.85 depending on the concrete element in the plastic truss model used to predict the capacity of continuous deep beams. Ashour [30] concluded that the best mean value of the effectiveness factor of reinforced concrete deep beams with fixed ends is 0.5. More values of the effectiveness factor proposed in codes of practice are presented later in section 5.2 of this paper.

Foster and Malik [31] evaluated different effectiveness factor formulae used in strut-and-tie models of nonflexural members such as deep beams, corbels and nibs. They concluded that effectiveness factor models based primarily on concrete strength are found to have poor correlation with test results of 135 nonflexural structural elements. They recommended that effectiveness factor models, that

account for the angle of the strut relative to the longitudinal axis of the structural member combined with models based on the modified compression field theory (Eq. 3 above), are found to give best correlation with the experimental results.

It is well documented that shear strength of reinforced concrete beams without web reinforcement appears to decrease as the beam depth increases [10, 32] and this size effect is less significant for beams with web reinforcement. Size effect could not be directly considered in the plasticity theory in which the nominal stress at failure must be independent on the size of structures. The only possibility to accommodate the effect of deep beam size would be to have the effectiveness factor depending on size. Yang et al. [33] modified the effectiveness factor proposed by Vecchio and Collins (Eq. 3 above) by a size effect factor which is a function of deep beam effective depth and maximum size of aggregate. They concluded that the capacity of continuous deep beams with different section depths was accurately predicted; indicating that size effect is successfully represented in the modified effectiveness factor. On the other hand, Tan and Cheng [34] suggested that size effect depends on factors such as the geometry of strut (width and length) and strut boundary conditions due to transverse web reinforcement (spacing and diameter). They proposed a modified strut and tie model that accurately predicts the size effect trends for deep beams, with a uniform safety margin for different member sizes considered.

5 LOWER BOUND ANALYSIS OF REINFORCED CONCRETE DEEP BEAMS

The lower-bound analysis of reinforced concrete deep beams is often developed from a hypothetical plastic truss model. In contrast to the situation for truly plastic materials, the validity of the chosen truss model for a reinforced concrete deep beam depends on whether the truss model represents the true situation reasonably close or not as reinforced concrete deep beams can undergo only a limited amount of redistribution of internal forces. Therefore, if the chosen truss requires excessive deformation to reach the fully plastic state assumed, the beam may fail prematurely at a load lower than that predicted by the truss. Extensive investigations have been carried out to improve the predictions from strut-and-tie models [1, 2, 9, 10, 29, 35-37]. Strut-and-tie models developed in the literature for different reinforced concrete deep beam cases are presented and discussed below.

5.1 Strut-and-Tie Models of Deep Beams

Diagonal cracks in the web area of deep beams separate the concrete into a series of diagonal concrete struts which are assumed to resist compression. The diagonal concrete struts in deep beams commonly considered as bottle-shaped struts that are generally idealised as prismatic or uniformly tapered members within shear spans [1, 2, 3, 10, 29, 36]. A tension tie represents one or several layers of steel reinforcement. Nodes are the joints where axial forces in struts and ties intersect. Marti [36] pointed out the importance of considering actual dimensions of compressive struts and tensile ties in formulating the truss models.

Fig. 2 illustrates schematic strut-and-tie models for reinforced concrete deep beams subjected to two-point symmetrical top loads, as suggested by different researchers: Fig. 2(a) for simply supported deep beams [1, 2, 9, 12-14, 16, 19] and Fig. 2(b) for continuous deep beams [18, 29]. The contribution of web reinforcement to the beam capacity is ignored. The strut-and-tie model of a single span deep beam shown in Fig. 2(a) is composed of two diagonal concrete struts, a top flexural concrete strut and a horizontal bottom steel tie connected together at four nodal zones. Nodal zones at the applied load point would be classified as a CCC type, which is a hydrostatic node connecting two concrete struts and external applied load, whereas nodal zones at supports are CCT type anchoring the bottom horizontal tie to the diagonal concrete strut and support bearing area. In a CCC type nodal zone having equal stresses on all in-plane sides, the ratio of each face width of the hydrostatic node has to be the same as the ratio of forces meeting at the node to make the state of stresses in the whole node region constant [9, 36, 38]. Concrete stress levels in nodal zones must be controlled to allow for the safe transfer of forces, which depends on many factors, including the tensile straining from tension ties, confinement provided by reactions and concrete compression struts, and confinement provided by transverse reinforcement. A diverse range of limits on concrete stresses in different nodal zones was suggested in the literature [9, 10, 38]. Yun [38] conducted an extensive review on the approaches for evaluating nodal zone strength. He concluded that all approaches checked the concrete compressive stress in the nodal zone boundary and proposed that non linear finite element analysis of the nodal zone must be conducted to accurately predict the limiting stresses within the nodal zone.

Considering equilibrium of forces, the load capacity P of simple deep beams shown in Fig. 2(a) owing to crushing failure of diagonal concrete strut is:

$$P = F_E \sin\theta \tag{5}$$

where F_E = load capacity of concrete struts and θ = angle between the concrete strut and the longitudinal axis of the beam, which can be expressed as $\tan^{-1}(jd/a)$, where a = beam shear span as shown in Fig. 2(a). The distance between the centre of top and bottom nodes, jd, in simple deep beams could be approximately assumed as the distance between the bottom reinforcement and the centre of the top node. The load transfer capacity F_E of concrete struts depends on the strut area and effective concrete compressive strength. Hence, the load transfer capacity F_E of concrete struts in simply supported deep beams is:

$$F_E = v f_c^{'} bw \tag{6}$$

where b and w are the beam and strut widths, respectively, as shown in Fig. 2(a). The strut width w is dependent on sizes of the tie and loading plate, and the slope θ of the diagonal compression strut of concrete. Other failure modes such as yielding or anchorage failure of longitudinal reinforcement, diagonal splitting failure of concrete and bearing failure at load or support plates may occur, but failure owing to crushing of concrete struts is the most common [19].

The strut and tie model for continuous deep beams shown in Fig. 2(b) is statically indeterminate. Few techniques [13, 18, 29, 39] were proposed to analyse statically indeterminate strut-and-tie models to determine the distribution of forces. One method is to assume the most heavily loaded ties yielded until the truss system becomes statically determinate [29]. Another technique is to decompose the statically indeterminate truss into several statically determinate trusses [18] and the third approach is to carry out stiffness analysis for the indeterminate truss [13, 39]. Therefore, the strut-and-tie model of the continuous deep beam illustrated in Fig. 2(b) could be subdivided into two main load transfer systems: one of which is the strut-and-tie action formed with the longitudinal bottom reinforcement acting as a tie and the other is the strut-and-tie action due to the longitudinal top reinforcement. The node at the applied load point is a combination of two types: a CCC type, which is a hydrostatic node connecting both exterior and interior compressive struts and bearing areas, and a CCT type node to anchor the longitudinal top reinforcement.

As the applied loads in the two-span continuous deep beams are carried to supports through concrete struts of exterior and interior shear spans, the load capacity P of continuous deep beams owing to failure of concrete struts is:

$$P = (F_E + F_I)\sin\theta \tag{7}$$

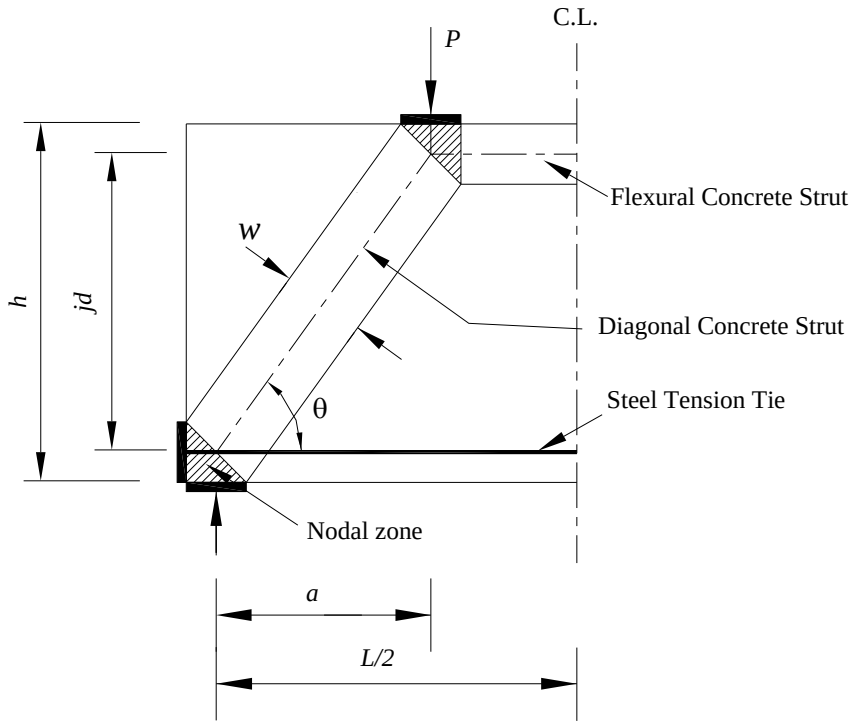
where F_E and F_I = load capacities of exterior and interior concrete struts, respectively, and θ = angle between the concrete strut and the deep beam longitudinal axis, which can be expressed as $\tan^{-1}(jd/a)$, where a = beam shear span as shown in Fig. 2(b). The distance between the centre of top and bottom nodes, jd, could be approximately assumed as the distance between the centre of longitudinal top and bottom reinforcing bars.

The load transfer capacities of exterior and interior concrete struts can be estimated as below, similar to that of simply supported beams (See Eq. 6):

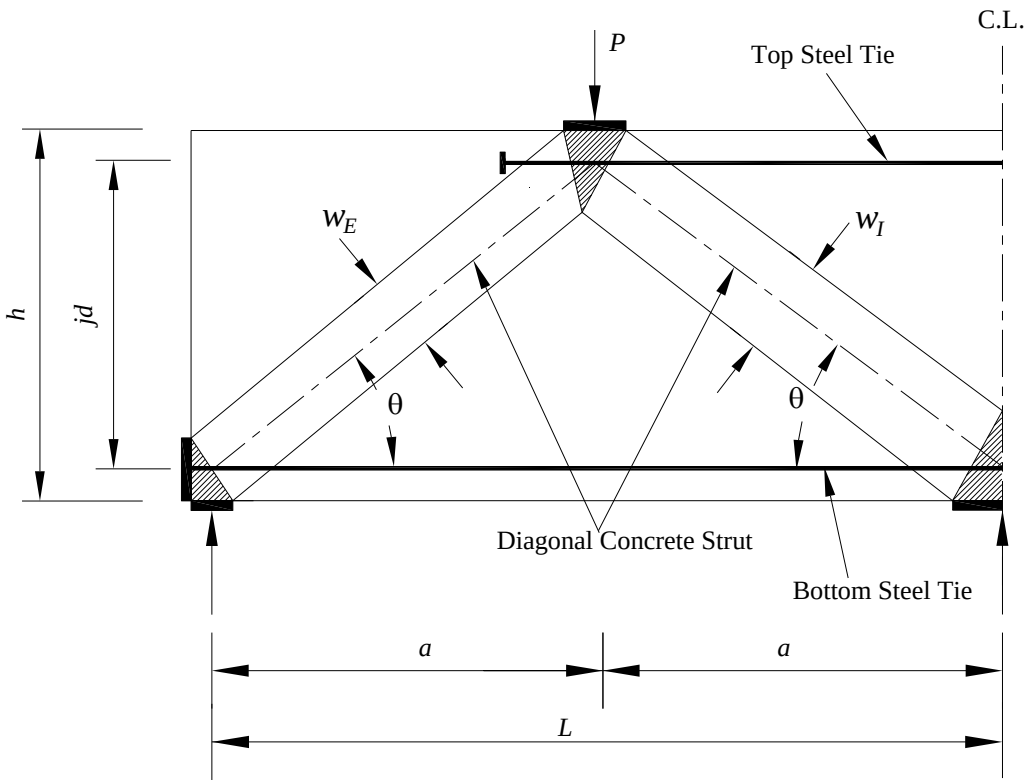
$$F_E = \nu f_c^{'} bw_E \tag{8}$$

$$F_I = \nu f_c^{'} bw_I \tag{9}$$

where w_E and w_I = widths of exterior and interior struts, respectively, as shown in Fig. 2(b).



(a) Simply supported beams



(b) Continuous deep beams

Fig. 2 Qualitative strut-and-tie models for reinforced concrete deep beams.

Rogowsky and MacGregor [29] presented plastic truss models shown in Figs. 3 and 4 to predict the capacity of continuous deep beams with either vertical or horizontal web reinforcement. For beams with horizontal web reinforcement, there are two trusses to transfer the applied loads to supports as depicted in Fig. 3. In this case, it is expected that the bottom reinforcement would reach yield before the upper horizontal steel reinforcement. However, the additional deformations required for the upper steel reinforcement to yield so that the upper truss can reach its full capacity would generally be large enough to cause beam collapse. Therefore, it was suggested that horizontal web reinforcement would be neglected in the truss model [29]. For deep beams with vertical reinforcement, the applied load is transferred to supports by direct diagonal struts and two compression fans radiating from the applied point load and support reaction as shown in Fig. 4.

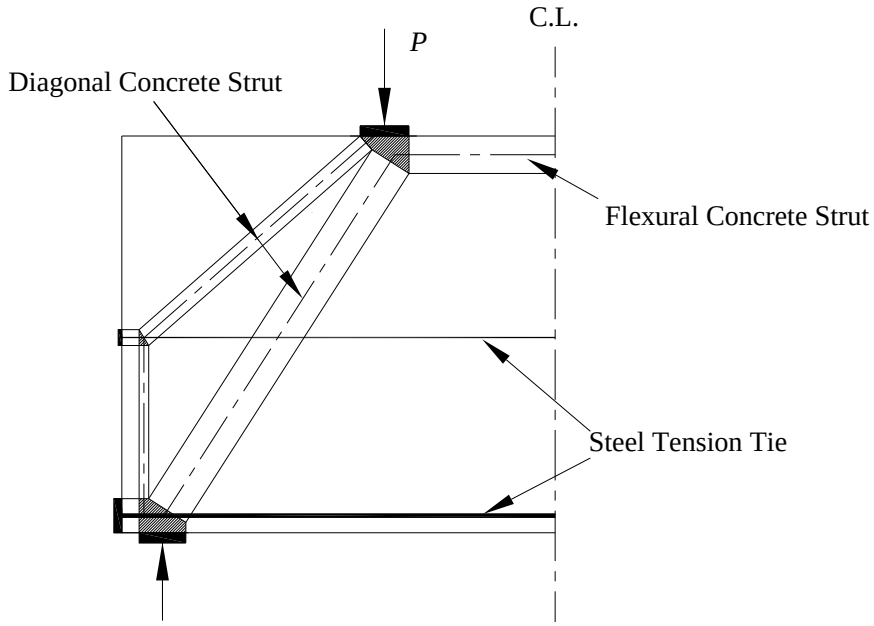


Fig. 3 Strut-and-tie model for deep beam with horizontal web reinforcement.

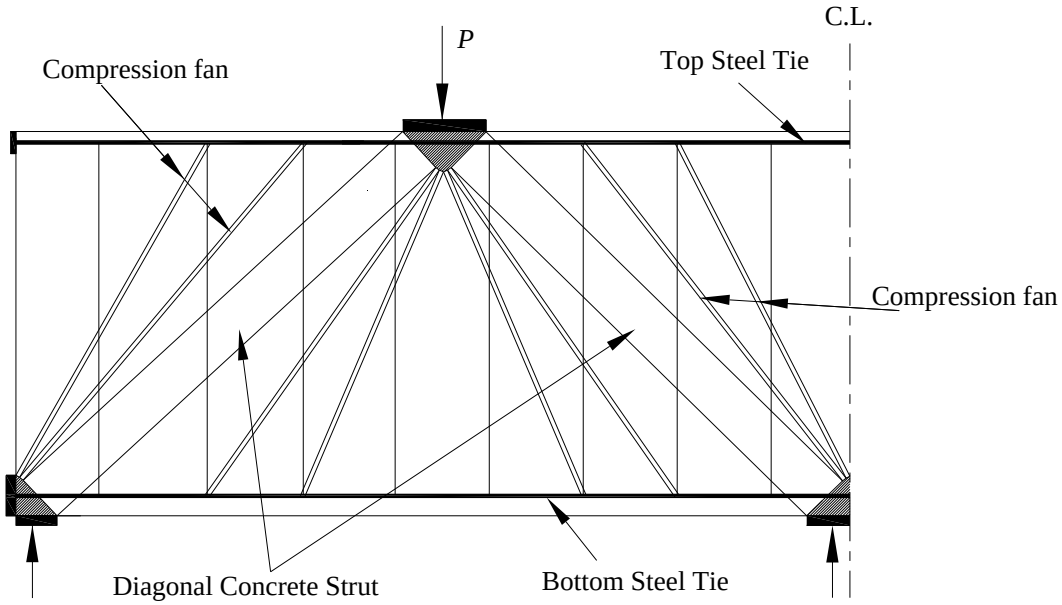


Fig. 4 Strut-and-tie model for continuous deep beam with vertical web reinforcement.

Other strut-and-tie models for reinforced concrete deep beams [12, 13, 40, 41, 42] were also proposed for estimating the load transfer mechanism of both horizontal and vertical web reinforcement and in case of deep beams with web openings. However, the detailing of such models considering the actual sizes of struts and ties proved to be complicated, therefore they were presented in a load path form.

Some research investigations [43, 44] have focused on automatic generation of optimal strut-and tie models using topology optimisation algorithms, especially for deep beams with web openings interrupting the follow of diagonal struts between applied loads and supports. The optimal strut-and-tie model is produced by gradually removing concrete regions that are ineffective in carrying loads based on overall stiffness performance criteria. Such techniques heavily utilise the finite element method as a modelling and analytical tool.

5.2 Code Modelling of Deep Beams

Earlier design provisions for reinforced concrete deep beams, for example ACI 318-99 [46] and CIRIA Guide 2 [47], adopted empirical design procedures to estimate the shear capacity of deep beams. However, most current codes of practice [3-6] recommend the use of strut-and-tie models for designing reinforced concrete deep beams. Nonetheless, they do not provide specific guidance on suitable strut-and-tie models for different deep beam cases. For example, no specific guidelines on the truss action identifying the load transfer mechanism of horizontal and vertical shear reinforcement and in case of web openings are provided.

Different effectiveness factors for concrete are proposed in codes of practice. AASHTO LRFD Specification [5] and CSA A23.3-04 [6] consider the effectiveness factor as a function of the amount of transverse tensile strain, whereas Eurocode 1992 [4] gives it as a function of concrete strength. On the other hand, ACI 318-05 [3] allows the use of effectiveness factor of 0.75 for concrete struts having a minimum amount of shear reinforcement, regardless of concrete strength and the amount of transverse tensile strain. The minimum amount of shear reinforcement required in bottle-shaped struts, which is recommended to be placed in two orthogonal directions in each face, is suggested by ACI 318-05 as follows:

$$\sum \frac{A_{si}}{b_w s_i} \sin \gamma_i \geq 0.003 \tag{10}$$

where A_{si} and s_i = the total area and spacing of the i–th layer of reinforcement crossing a strut, respectively, and γ_i = the angle between the i–th layer of reinforcement and a strut. This minimum reinforcement contributes significantly to the ability of a deep beam to redistribute the internal forces after cracking. The value of the effectiveness factor drops to 0.6 if the minimum shear reinforcement defined in Eq. (5) above is not provided. This implies that the arrangement of shear reinforcement satisfying Eq. (5) allows the load capacity of deep beams predicted by the strut-and-tie model to be increased by 25%.

ACI 318-05 stipulates the concrete effective stresses in the nodal zones shall not exceed a certain value: $0.85 f_c^{'}$ in CCC type nodes bounded by compression struts and bearing areas as shown in the top node of Fig. 2(a), and $0.68 f_c^{'}$ in CCT type nodes anchoring only one tension tie as shown in the bottom nodes of Fig. 2. However, Eurocode 1992 proposed that the concrete effective stresses in CCC and CCT type nodes are limited to $v f_c^{'}$ and $0.85 v f_c^{'}$, respectively, where v is the effectiveness factor for concrete struts.

Most codes of practices limit the angle between concrete struts and steel ties joined at the same nodal zone. ACI 318-05 limits this angle to values greater than 25 degrees, whereas Eurocode 1992 recommends limiting values for the angle of the inclined struts in the web to be between 22.5 and 45 degrees.

6 UPPER BOUND ANALYSIS OF REINFORCED CONCRETE DEEP BEAMS

The first upper bound analysis for shear strength of reinforced concrete beams was developed by Nielsen and his associates in Denmark [8]. Kemp and Al-Safi [48] derived an upper bound solution for reinforced concrete beams based on rotation and translation of rigid blocks. The first attempt to generalise the upper bound analysis for plane stress problem was made by Zainai and Morley [49, 50], when they derived upper bound solutions for deep beams with web openings considering more than one yield line.

6.1 Failure Mechanisms of Deep Beams

The failure mechanism of reinforced concrete deep beams is idealised as an assemblage of rigid blocks moving in the beam plane, separated by yield lines. The yield line is a theoretical representation of the narrow discontinuity zone with many criss-crossing cracks and crushing zones which occurs in reality [7, 8, 22, 25]. It is used not in the sense of flexural yield lines in slabs but as lines along which in-plane displacement discontinuities take place. It was proved [50, 51, 52] that the optimum shape of the yield line is a hyperbola as the energy dissipated along it is less than that dissipated in a straight yield line. This hyperbolic yield line turns into two straight segments when the instantaneous centre (I.C.) of relative rotation of rigid blocks lies inside or on the circle whose diameter is the straight line between the end terminals of the yield line. As a special case, when the instantaneous centre (I.C.) approaches infinity, the hyperbolic yield line reduces to a straight yield line between the edges of point load and support plates. In this case, there is a pure translation of rigid blocks relative to each other as considered earlier by Nielsen et al. [8, 22, 25].

Figs. 5 to 9 shows idealised mechanisms of failure of reinforced concrete deep beams with different end conditions and with or without web openings studied in the literature [8, 11, 17, 18, 30, 52, 53]. These mechanisms were experimentally observed at failure.

The unsymmetrical mechanism of simply supported deep beams under two symmetrical vertical loads shown in Fig. 5(a) consists of two rigid blocks moving in the beam plane, separated by a yield line. Whereas the symmetrical failure mechanism of simply supported deep beams shown in Fig. 5(b) would be idealised as an assemblage of three rigid blocks separated by two yield lines [8, 11, 17, 52].

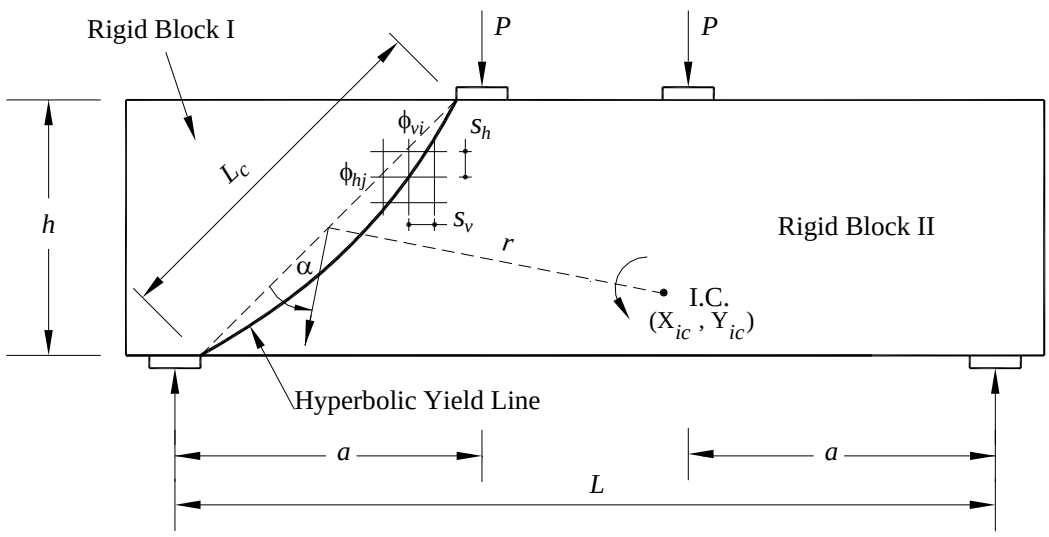
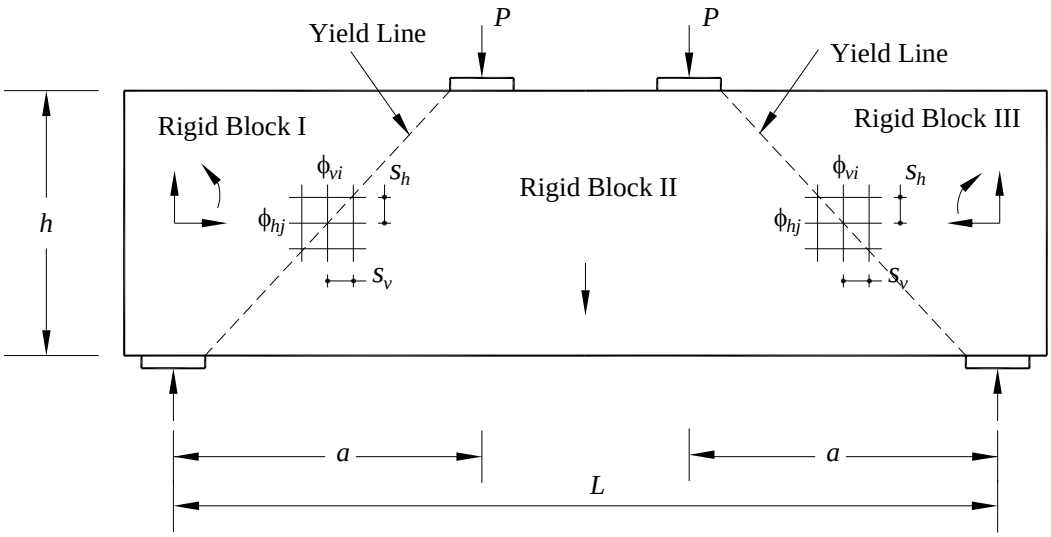


Fig. 5 (a) Un-symmetrical Mechanism



(a) Symmetrical Mechanism

Fig. 5 Failure mechanisms of simply supported deep beams.

At collapse, continuous reinforced concrete deep beams failing in shear can usually be idealised as an assemblage of two rigid blocks separated by a yield line as shown in Fig. 6. This is more likely to occur at the end span of continuous beams [18, 26]. Another failure mechanism for continuous deep beams, shown in Fig. 7, is a pure translation or diagonal splitting mechanism without rotation [26, 30]. It consists of three blocks separated by two yield lines. The central block is moving vertically, whereas the other two blocks are fixed. This mechanism would occur when the beam ends are fully continuous or fixed, or when heavy horizontal reinforcement is used and prevents block rotation.

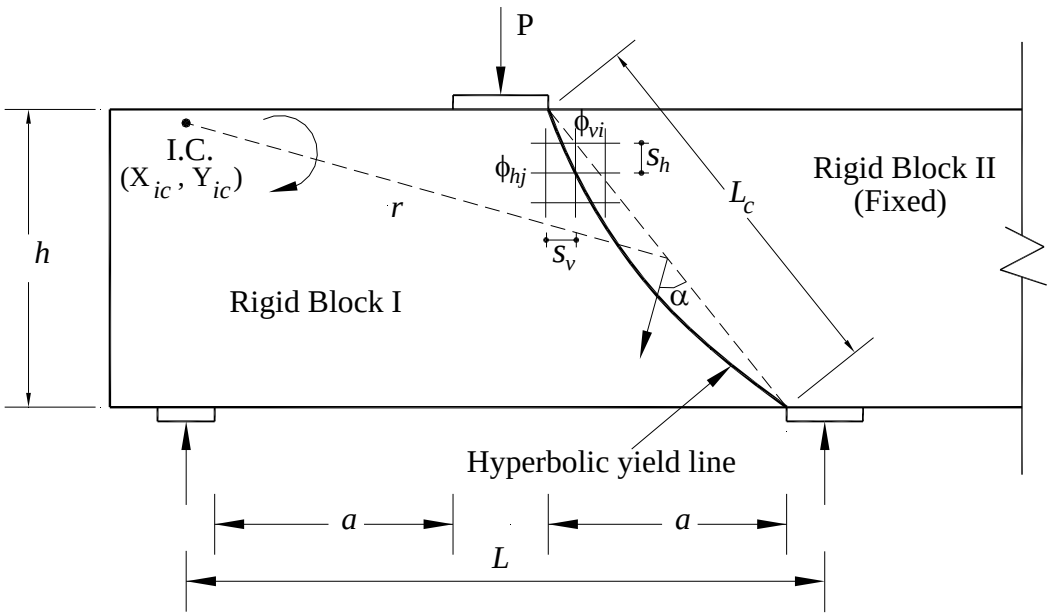


Fig. 6 Failure mechanism of end span continuous deep beams.

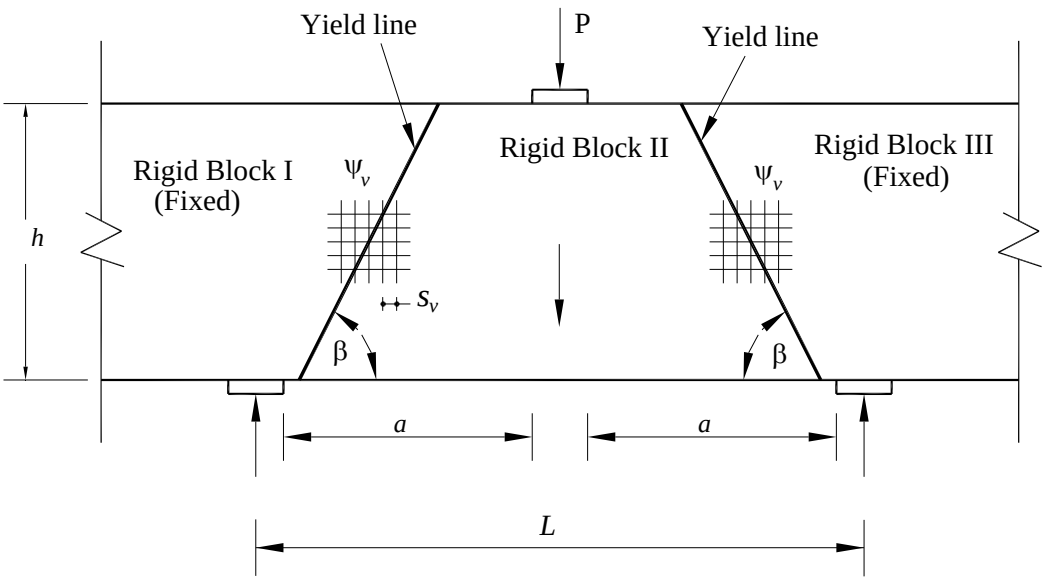


Fig. 7 Diagonal splitting (pure translation) failure mechanism.

Fig. 8 presents a combined flexure and shear failure mechanism that would occur in deep beams with fixed end supports [30]. This mechanism is idealised as an assemblage of two rigid blocks separated by two yield lines as shown in Fig. 8. The flexural yield line occurs along one of the fixed end supports, and the diagonal shear yield line takes place in the shear span opposite to the flexural yield line.

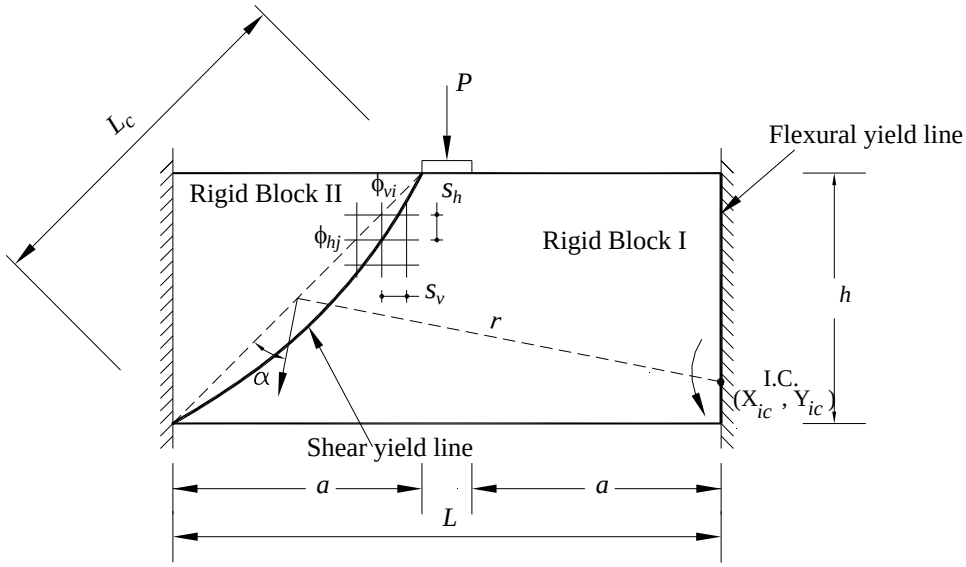
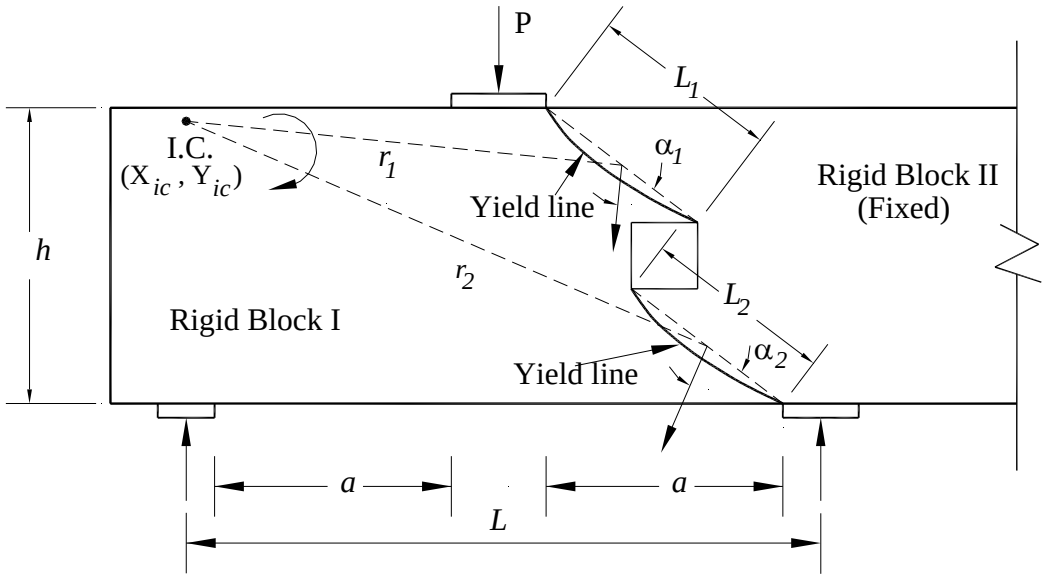
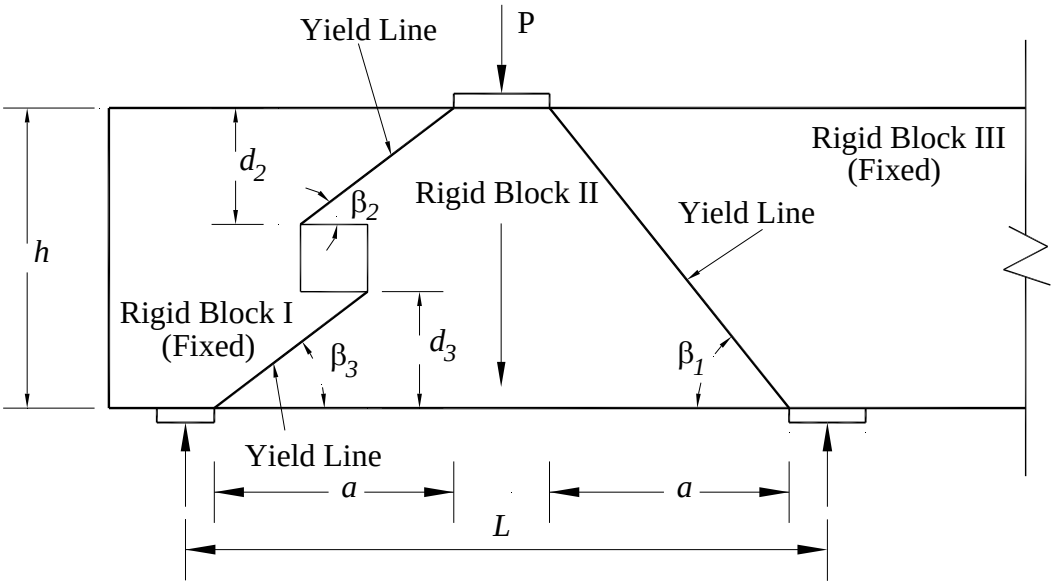


Fig. 8 Failure mechanisms (Flexure and shear) of deep beams with fixed end supports.

The mechanism of failure that investigated for continuous deep beams having web openings within interior shear spans of beam end span can be idealized as an assemblage of two rigid blocks separated by two yield lines [53], as shown in Fig. 9(a). Fig. 9(b) presents an idealisation of the failure mechanism observed for continuous deep beams having web openings within exterior shear spans [53]. It consists of three rigid blocks separated by three yield lines. These two mechanisms are, in principle, similar to those of continuous deep beams without web openings (see Figs. 6 and 7), but the presence of web openings interrupts the continuity of yield lines. Zainai and Morley [49, 50] and Ashour and Morley [52] presented similar mechanisms for simply supported deep beams with web openings.



(a) Beams having Web Openings within Interior Shear Span



(b) Beams having Web Openings within Exterior Shear Span

Fig. 9 Failure mechanisms of continuous deep beams with web openings.

6.2 Upper Bound on Load Capacity of Deep Beams

In general, each rigid block formed at failure in the above mechanisms has two transitional and one rotational displacement components. Considering boundary conditions at supports and/or beam symmetry, the independent displacement components for each block would be reduced.

The load capacity of deep beam mechanisms presented above is obtained by equating the total internal energy dissipated in concrete and steel reinforcement along yield lines to the external work done by applied loads. All steel reinforcing bars crossing yield lines are assumed to be yielded. Although, the energy dissipated depends on the rigid block displacements and the location of the instantaneous centre (I.C.) of relative rotation of blocks, the load capacity obtained for each mechanism is generally expressed as a function of the concrete and steel properties, and position of the instantaneous centre, (X_{ic}, Y_{ic}). Table 1 presents the normalised load capacity, $\lambda = P/(bhf_c^{'})$, of deep beams with different end conditions, and with or without web openings obtained from the mechanism approach.

According to the upper-bound theorem of the plasticity theory, the collapse occurs at the least strength. The minimum value of the load capacity is obtained by varying the position (X_{ic}, Y_{ic}) of the instantaneous centre in the vertical plane of the deep beam. This is normally achieved by numerical optimisation techniques. Alternatively, if main longitudinal steel bars are sufficiently strong not to yield, the instantaneous centre of rotation would be located at their level (i.e., steel does not yield). In deep beam cases where more than one mechanism of failure would occur, the governing (guide) mechanism would be the one predicting the lowest capacity.

Table 1 Normalised load capacity λ of deep beams predicted by mechanism analysis.

Source	Deep beam details	Mech-anism	Normalised load capacity, $\lambda = P/(bhf_c^{'})$
[11]	Simply supported deep beams	Fig. 5(a) & Fig. 5(b)	$\lambda = \frac{1}{X_{ic}}\left[\frac{\nu}{2h}L_c r(1-\sin\alpha) + \sum_{i=1}^{N_{\phi v}}\phi_{vi}\left\lvert X_i - X_{ic}\right\rvert + \sum_{j=1}^{N_{\phi h}}\phi_{hj}\left\lvert Y_j - Y_{ic}\right\rvert\right]$
[18, 26]	Continuous deep beams	Fig. 6	$\lambda = \frac{1}{L/2}\left[\frac{\nu}{2h}L_c r(1-\sin\alpha) + \sum_{i=1}^{N_{\phi v}}\phi_{vi}\left\lvert X_i - X_{ic}\right\rvert + \sum_{j=1}^{N_{\phi h}}\phi_{hj}\left\lvert Y_j - Y_{ic}\right\rvert\right]$
		Fig. 7	$\lambda = \nu\left[\sqrt{1+(\frac{a}{h})^2} - \frac{a}{h}\right] + 2\psi_v\frac{a}{h}$
[30]	Deep Beams with Fixed End Supports	Fig. 8	$\lambda = \frac{1}{L/2}\left[\frac{\nu}{2h}[L_c r(1-\sin\alpha) + Y_{ic}^2] + \sum_{i=1}^{N_{\phi v}}\phi_{vi}\left\lvert X_i - X_{ic}\right\rvert + \sum_{j=1}^{N_{\phi h}}\phi_{hj}\left\lvert Y_j - Y_{ic}\right\rvert + \sum_{k=1}^{N_{\phi h1}}\phi_{hk}\left\lvert Y_k - Y_{ic}\right\rvert\right]$
[53]	Continuous deep beams with web openings	Fig. 9(a)	$\lambda = \frac{1}{L/2}\left[\frac{\nu}{2h}[L_1 r_1(1-\sin\alpha_1) + L_2 r_2(1-\sin\alpha_2)] + \sum_{i=1}^{N_{\phi v}}\phi_{vi}\left\lvert X_i - X_{ic}\right\rvert + \sum_{j=1}^{N_{\phi h}}\phi_{hj}\left\lvert Y_j - Y_{ic}\right\rvert\right]$
		Fig. 9(b)	$\lambda = \frac{\nu}{2}\left[\frac{1-\cos\beta_1}{\sin\beta_1} + \frac{d_2}{h}\frac{1-\cos\beta_2}{\sin\beta_2} + \frac{d_3}{h}\frac{1-\cos\beta_3}{\sin\beta_3}\right] + \sum_{i=1}^{N_{\phi v}}\phi_{vi}$

Note: $\lambda = (P/bhf_c^{'})$ = dimensionless load; $\phi_{vi} = (A_{vi} f_{yi}/bhf_c^{'})$ and $\phi_{hj} = (A_{hj} f_{yj}/bhf_c^{'})$ = vertical and horizontal reinforcement ratios for individual bars crossing the yield line, respectively; A_{vi}, X_i, and f_{yi} = area, horizontal coordinate, and yield strength of the vertical bar *i* crossing the yield line, respectively; A_{hj}, Y_j, and f_{yj} = area, vertical coordinate, and yield strength of the horizontal bar *j* crossing the yield line, respectively; $N_{\phi v}$ and $N_{\phi h}$ = number of vertical and horizontal bars crossing the hyperbolic yield line; the subscript *k* is used to identify the horizontal reinforcement of total number $N_{\phi h1}$ crossing the flexural yield line in case of deep beams with fixed end supports; $\psi_v = (A_{vi} f_{yi}/bs_v f_c^{'})$ = smeared intensity of the vertical shear reinforcement, and s_v = vertical web reinforcement spacing. The rest of geometrical notations are defined in different figures above.

7 PREDICTIONS OF DEEP BEAM CAPACITY USING PLASTICITY THEORY

The accuracy of predictions of deep beam capacity obtained from both strut-and-tie models and mechanism analysis was studied in many investigations [11-19]. Table 2 presents summary of statistical parameters of the comparisons between the predicted deep beam capacity and experiments. Comparisons conducted for large database of deep beams are only presented in Table 2. However, most validations were carried out for simply supported deep beams without web openings. The statistical parameters show the efficiency of both the strut and tie models and mechanism analysis in predicting the deep beam capacity. In addition, the plasticity-based analyses of deep beams accurately predicted the trend of the deep beam capacity against different influencing parameters, for example, the decrease of the deep beam capacity with the increase of the shear-span-to-depth ratio [8, 10, 11, 26] as experimentally observed in many investigations.

Table 2 Comparisons between experimental and predicted load capacities.

Source	Supporting condition	No. of beams	μ^*	σ^*	ρ^*
(a) Strut-and-tie models					
[12]	Simple	123	1.15	0.16	0.14
[13]	Simple	175	1.40	0.31	0.22
[14]	Simple	240	1.00	0.19	0.19
[15]	Simple	51	1.04	0.12	0.12
[16]	Simple	233	0.76	0.29	0.38
[19]	Simple	448	1.31	0.47	0.36
[18]	Continuous	75	1.12	0.37	0.33
(b) Mechanism approach					
[17]	Simple	64	1.02	0.13	0.13
[11]	Simple	172	1.02	0.20	0.20
[18]	Continuous	75	1.07	0.14	0.13

* μ, σ and ρ stand for the mean, standard deviation and coefficient of variations of the ratio between the experimental and predicted load capacities. All deep beams considered in the above studies were solid, without web openings.

8 CORRELATION AND CONCLUSIONS

The following correlation and conclusions are deduced from the survey on the application of plasticity theory to reinforced concrete deep beams presented in this paper:

- Upper and lower bound analyses are powerful tools for predicting load capacity of reinforced concrete deep beams. Strut-and-tie models give a better understanding of the distribution of internal forces within deep beams, whereas upper bound analysis would encourage designers to examine different failure mechanisms of deep beams. Compared with methods based on empirical or semi-empirical equations, upper and lower bound analyses are more rational, adequately accurate and sufficiently simple for reinforced concrete deep beams.
- Strut-and-tie models received more attention in recent years than the mechanism analysis. This would be attributed to the fact that plasticity-based strut-and-tie models theoretically produce safe, lower bound designs. However, mechanism analysis would also produce a conservative prediction when the effectiveness factor is carefully selected.
- Although more than one admissible strut and tie model for deep beams can be developed, designers should be careful that the chosen model would represent the true situation reasonably close as reinforced concrete deep beams have a limited ductility. On the other hand, different mechanisms of failure have to be examined before identifying the deep beam capacity.
- There is a problem of selecting the effectiveness factor of concrete as reflected in the wide range of values reported for deep beams in the literature.
- The plasticity models for deep beams assume that horizontal and vertical steel bars crossing failure zones have yielded. An assumption, which has not been experimentally fulfilled in case of shear failure of reinforced concrete deep beams, significantly contributes to inconsistent predictions in few cases.
- The detailing of the strut-and-tie model is strongly influenced by the size of the loading plates and the location of main longitudinal reinforcement. In addition, the main longitudinal reinforcing bars in deep beams are not carried through to anchor plates as the stress fields pretend.
- Strut-and-tie models are generally more difficult to develop for deep beams with orthogonal web reinforcement or web openings than the mechanism approach.

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