WHEN PLASTICITY?

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1 BACKGROUND

The need for adequate ductility in practical concrete structures is widely recognised, to cope with various circumstances, especially in indeterminate structures – and the appropriate theory for the collapse load of ductile structures, short of non-linear finite-element or similar computerised analysis, is *plasticity theory.* Many concrete structures are in practice designed using, for the Ultimate Limit State of collapse, calculations based on some form of plasticity theory, perhaps with modifications - with serviceability achieved by rules on such things as span/depth ratio or bar spacing.

The *lower-bound* or *equilibrium* approach to design for a given ultimate load, or the concept of designing for a chosen load path without needing to know the actual path adopted in the structure, came in early on, e.g. in 1920's flat-slab design and the truss analogy for shear. It features in modern design codes in various places, e.g. in neglect of 'compatibility torsion', in ULS design of frames, and in allowing 'redistribution of moment' in continuous beams (which perhaps obscures the ductility desirable even when designing for the load path according to linear-elastic theory based on the gross concrete section).

Upper bound or mechanism approaches also came in early for reinforced concrete structures, e.g. in the work of Ingerslev [1] and Johansen on slabs [2], long before the well-known plasticity theorems for ductile metal structures were fully formulated in the 1950's. These approaches also appear in codes of practice, e.g. in the permitted use of yield line theory for assessment of slabs.

Attempts have been made to apply plasticity concepts to other concrete structures beyond plane frameworks and slabs in flexure, for example to shear failure of ordinary and deep beams, to voided slabs and box girders, to punching shear in slabs. Some of these applications have taken hold, e.g. the strut-and-tie method for designing details and the ultimate pure torque of some reinforced beams.

A comprehensive exposition of plasticity theory for concrete structures, as well as its historical development, is given in the mammoth work by Nielsen [3], now in a much-revised second edition [4].

Design for shear in beams according to the recent Eurocode [5] is now, when steel shear reinforcement is needed, fully based on ideas from plasticity theory, in particular the variable-angle truss analogy (a lower-bound approach) – sweeping away earlier ideas of adding a 'steel contribution' to a 'concrete contribution' mysteriously the same as the strength of the unreinforced concrete web (which depends on concrete tensile strength) despite the fact that extensive cracking occurs before the steel stirrups develop full strength.

However, some attempts to apply plasticity theory to concrete structures have not caught on in practice, especially where the behaviour of the concrete dominates, and concrete's inadequate ductility, in compression as well as tension, becomes manifest, e.g. in punching shear in slabs. Here there is much uncertainty about the 'effectiveness factor', less than unity and often rather unpredictable, which has to be applied to the strength measured on concrete control specimens to get theory to agree with the results of tests on structures.

The requirement for some ductility in robust practical structures perhaps needs re-emphasis, in the face of rumoured trends towards the use of lower-ductility steel for reinforcement, and (especially) the use of carbon-fibre-reinforced-plastic (CFRP) – strong but very brittle – as reinforcement, particularly to strengthen existing concrete structures thought weak for some reason. However, one may well question the application of plasticity theory, with its heavy reliance on redistribution of stress as load increases, to structures in such low-ductility materials – even if the theory is heavily festooned with effectiveness factors and has ample allowance for size effects.

This paper considers, without attempting to cover the whole field, some queries that have been raised about the application of plasticity theory to concrete, and one case where plasticity theory for slabs, and in particular the lower-bound theorem, has perhaps been pushed too far. The paper goes on to describe some recent work in Cambridge on concrete slabs with short steel fibres added to the mix, thus increasing the ductility in tension, to investigate whether a 'purer' form of plasticity theory, for bending or punching shear, might then become appropriate. The overall aim is to raise the question of when plasticity theory may justifiably be applied to concrete structures.

2 WEAKENING DUE TO CRACKING

It is well known that plain concrete normally fails by cracking of some kind, showing very little ductility in tension; that in uniaxial compression there is no plastic plateau but non-linear behaviour (Fig.1) with a falling branch (and cracks roughly parallel to the stress); but that in triaxial (and to a much lesser extent, biaxial) compression concrete shows increased strength and rather greater ductility. In many beams, plane frames and slabs the ductility required for plastic theory is provided by the main steel reinforcement, often with limited (and arguably beneficial) bond slip near cracks. This has led to the successful development of a modified form of plastic theory, based on behaviour at the stress-resultant level (moments etc) with checks on rotation capacity in hinges, often related to the percentage of reinforcement. However, when attempts are made to apply plasticity theory in other situations, with assumed yield surfaces for the individual materials, difficulties can arise.

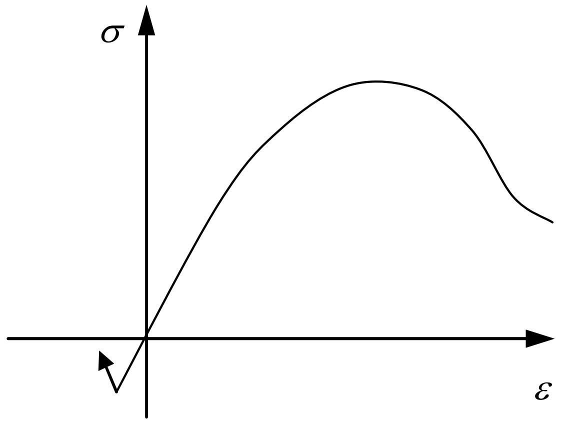

Fig. 1 Uniaxial stress-strain curve for typical concrete

One objection to the use of plasticity theory for concrete can perhaps best be regarded as querying the meaning of the yield surface commonly plotted for the material on principal stress axes (Fig.2), especially for the repeated loading cases inevitable in real structures. In metals, where plastic flow occurs by slip of layers of atoms, leaving the structure essentially undamaged – and perhaps even strengthened by work hardening due to interference of dislocations – the yield surface is at worst unchanged by plastic flow, and might be expanded by it. So design for one loading case can proceed without much worry about any damage to the material caused by other loading cases – though one might be concerned about incremental collapse.

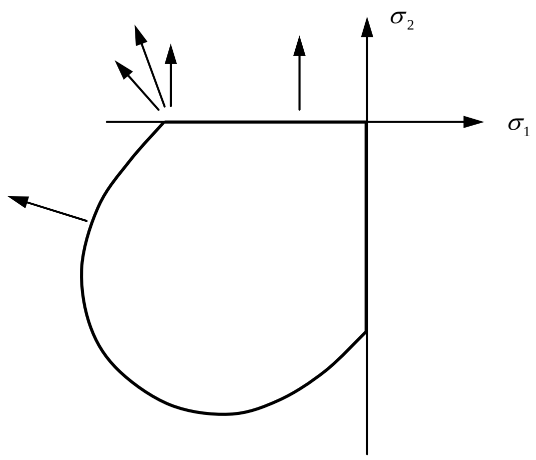

Fig. 2 Typical assumed yield surface for concrete under biaxial stress

In concrete on the other hand, ’plastic flow’ upon reaching the yield surface, especially in the tension region, occurs by opening of cracks, which are obvious sources of weakness and may permit, for example, frictional shear slip on the crack surface at lower strength than the original yield surface would predict. Thus we have the Vecchio and Collins [6] softening of struts in sheared concrete webs, with the compressive strength in the strut reduced by transverse tensile strain corresponding to the opening of cracks parallel to the compression – a phenomenon not easily represented in plastic theory, simple or otherwise.

Then there is the possibility, raised by Bazant [7] that variable and repeated loading cases in slabs leave the concrete criss-crossed by multiple cracks in several directions, effectively destroying its cohesion and reducing it to a granular material. This would need to have compression in all directions in order to develop any shear strength by friction, thus much increasing the reinforcement needed for a given loading case, so as to develop tension to balance this required compression. One might argue that if this is an important phenomenon it ought to be considered in any case of such multiple loading, whatever theory is being used to predict the strength of the structure – but it does seem to raise queries for plastic theory and concrete.

There is much resonance here with the revised book by Nielsen [4], which contains extensive discussion of the effect of cracking on strength – multiple microcracking possibly reducing cohesion, sliding on macrocracks with strength dependent on crack width relative to such things as aggregate size – all tending to lead us away from simple yield surfaces for the materials.

3 ‘DEFORMATION’ OR ‘FLOW’ THEORY

What happens when a material starts to unload, having undergone some plastic flow? There was much discussion of this for metals in the 1950’s and 1960’s [8,9], in the context of continuing load-deflection behaviour of a structure beyond its initial collapse load, as geometry-change becomes significant. In a uniaxial tension test beyond yield, if the total strain stops increasing and begins to reduce, the material immediately unloads elastically and plastic flow ceases: if the unloading proceeds so far that yield in compression occurs, compressive plastic flow begins, and the total accumulated plastic strain then begins to reduce. Incorporating this effect, in which the sign of the yield stress occurring depends on the sign of the current increment of plastic flow (and not the total accumulated plastic strain) gives rise to ‘flow theory’. By contrast, in ‘deformation theory’ the (non-linear but scarcely plastic) material is taken to unload back down the original stress-strain curve (Fig.3), so that the sign of the stress depends on the sign of the total accumulated plastic strain. In many ways flow theory fits better with the concepts of plasticity theory – the flow rule associated with the yield surface, discussion of plastic strain increments etc. – and papers were written, e.g. by Morley, Braestrup and Janas applying flow theory, in preference to deformation theory, to large-deflection behaviour of reinforced concrete slabs [10, 11, 12].

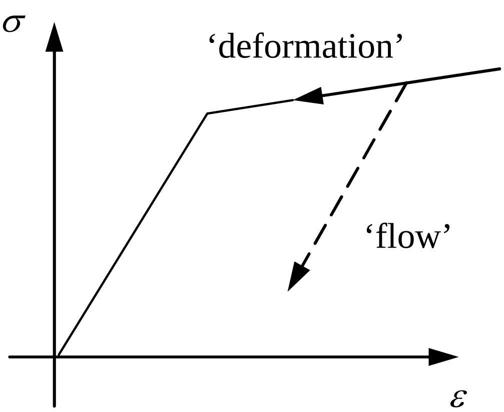

Fig. 3 Assumed loading and unloading curves for metals

The trouble is that for concrete’ flow theory’ does not always fit the facts well, as became clear when compressive membrane action in slabs was further investigated. ‘Plastic flow’ in tension occurs by the opening and widening of cracks – so that when the direction of strain is reversed the material cannot immediately go elastically into compression, as flow theory would predict. Rather the cracks have to close up, i.e. the tensile plastic strain must be eliminated rather than just reducing, before the material can go into compression. Thus deformation theory is more appropriate. This was pointed out by Kemp and Eyre [13, 14] who advocated using deformation theory for concrete unloading from tension, and flow theory for concrete unloading from compression. A very successful theory emerged for compressive membrane 'arching' action in laterally-restrained beams and slabs, where the crucial question is whether as deflection proceeds the neutral-axis is moving into or away from the concrete compression zone (requiring deformation or flow theory respectively). Applying such an approach to geometry beyond uniaxial, even to axisymmetric circular slabs, can become complicated [15] – and in any case is well beyond the concepts and theorems of simple plastic theory.

4 SIZE EFFECT

The next, and perhaps crucial, question for plasticity theory and concrete structures is that of 'size effect'. Suppose that one builds a series of structures of the same material(s) with a typical yield strength σ_y, of exactly the same geometrical shape but in different sizes, say with typical span L. The structures are then tested with the same boundary conditions on displacement, and the same loading pattern – and are found to collapse (with fairly small deflections) at total loads W_c . According to plastic theory, the ratio $W_c/\sigma_y L^2$ should be constant right across the series of structures – on dimensional arguments, there seems to be no other variable that this ratio could depend upon.

However, it is well known that if such tests are carried out for concrete structures, the ratio is not constant but tends to fall appreciably for structures of larger size L – there is a definite 'size effect', the more marked the more the collapse load depends on the properties of the concrete rather than the reinforcement. So what is to be done? Must plasticity theory be abandoned altogether? – which would be rather awkward in view of its apparently successful use over a wide range of practical structures of ordinary size. Even in Nielsen's second edition [4] there is comparatively little about size effect, though there is a warning to take the effect very seriously for dimensions substantially larger than those of normal laboratory specimens when the tensile strength is significant. In Nielsen's book size effect is mainly discussed in terms of Weibull theory, which is essentially about initiation of macrocracks and the increased chance of finding a critical flaw in a larger volume of concrete, rather than the possible sudden extension of an existing macrocrack which has grown to the same order of size as a leading specimen dimension (say the depth). According to Nielsen, size effect on the compressive strength may be neglected.

It seems possible that size effect is related to the suddenness or lack of ductility in a failure mode, so that unloading of elastic strain energy into a progressive failure can come into play, as in fracture mechanics concepts. If true, this would suggest that if different-size concrete structures could be reinforced such that all their failure modes were ductile – which means no unloading of elastic energy during collapse – there would be no size problem (just another way of saying that plasticity theory admits no size effect?).

The question for an engineer would then be how to know that all the failure modes of a large structure as designed would in fact be adequately ductile. Otherwise the comfort afforded by the lower bound theorem of plasticity could be illusory – whatever checks on such things as rotation capacity had been carried out. And what amount of ductility is adequate for this purpose? – investigations of the reduction in failure load related to various measures of ductility have been carried out, e.g. by Denton [16].

Alternatively, is plasticity theory, with all its interesting aspects and advantages, to be limited to comparatively small-size concrete structures, and for larger structures always supplemented by some form of size-effect law such as that due to Bazant [17] ?

5 TORQUELESS GRILLAGES

At the beginning of his Chapter 6 on slabs, Nielsen says 'plastic theory of reinforced concrete slabs is now rather old' [4, p.473]. Despite that, it might be interesting to present some recent work on torqueless grillages – grillages of closely-spaced beams, aligned with the orthogonal reinforcement directions in a slab and assumed to have no strength in torsion. The objective in considering these 'Hillerborg strips' [18] is usually to obtain satisfactory layouts of reinforcement for design of a slab against collapse, relying on the lower bound theorem of plasticity theory (and taking advantage of the fact that reinforcing against torque uses more steel).

One question is how to treat a very concentrated point load, such as might occur at a small-diameter column supporting a flat slab. As an example we consider an isolated rectangular zone with free edges under uniform applied load, supported on a central column. It turns out to be possible [19] to invent an equilibrium system, with shear forces and bending moments but no torsion, in which the load is carried inwards along the beam direction until it reaches a diagonal of the slab. At the diagonal there is a coincident jump in bending-moment magnitude in both strips intersecting at that point – and beyond the diagonal the bending moment in a strip remains constant (Fig.4).

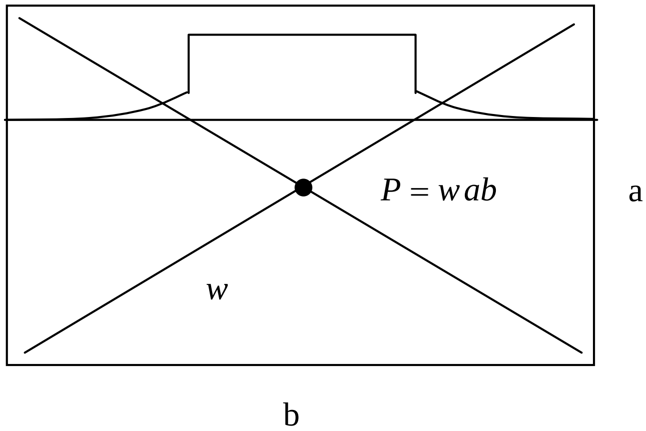

Fig. 4 Isolated rectangular zone on a central column – uniform load

From this point, the transverse force is carried inwards along the diagonal (a 'line of discontinuity') to the column – so that the solution can only be valid, and the lower bound theorem relied upon, if the slab can in fact take such high transverse forces along very narrow regions. As in most plastic theory of concrete slabs, there is an assumption that shear failures do not occur – and this assumption is being pushed especially far in this proposal. The proposed equilibrium system suggests that stud-rails and similar shear reinforcement should be placed along the diagonals of such near-column regions, rather than be aligned with the main reinforcement directions as is more usual – but the question arises whether, even with such shear reinforcement, the lower-bound theorem of plasticity can indeed be relied upon in practice in such a case. (Incidentally, it is definitely the lower-bound theorem that comes into question – the upper bound theorem remains true, if of less value, even in the face of reduced ductility, intervening shear failure etc.)

Further and perhaps more useful development of the torqueless grillage theory can be made. For example, long concrete bridges often have side cantilevers, carrying footway and cycle track but perhaps subject to heavy wheel loads from vehicles, in emergency or otherwise. Such a cantilever might have a tapering depth, and perhaps an edge beam. It would normally have fairly light reinforcement top and bottom running along the length of the bridge, and heavier top steel across the width of the cantilever. Some sagging strength across the width would be available for live load, by relieving the hogging cantilever moments due to self-weight (at its minimum value). So a possible model, for a point wheel load P placed anywhere within a certain distance of the cantilever root, would be as shown in Fig.5 - a very long slab of finite width, with four uniform bending strengths. For a lower bound solution, this could be modelled as a torqueless grillage of Hillerborg strips.

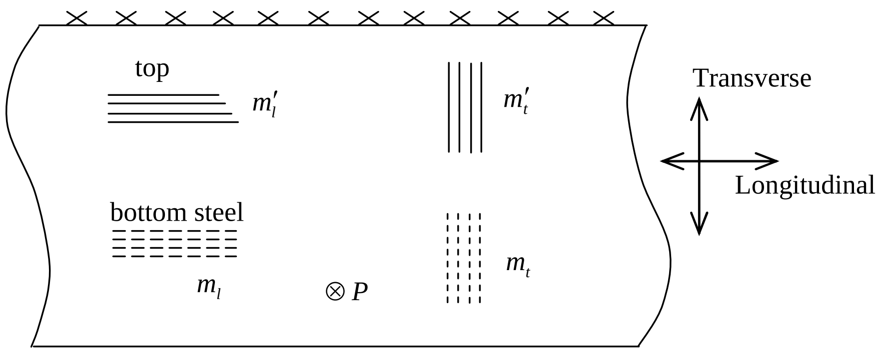

Fig. 5 Model of a long bridge cantilever, under point load P

An explicit *lower bound* on the collapse value of the point load was obtained by Lu [20] by considering discontinuity lines emanating from the load position, with jumps in the strip bending-moment diagrams as shown in Fig.6, and writing equilibrium equations for a typical small element on the discontinuity line with jumps in moment across it. This gives the required angle of the discontinuity lines, and the lower bound P_L on the collapse load is given by

$$P_L^2 = 4 m_t'(m_l + m_l') \quad (1)$$

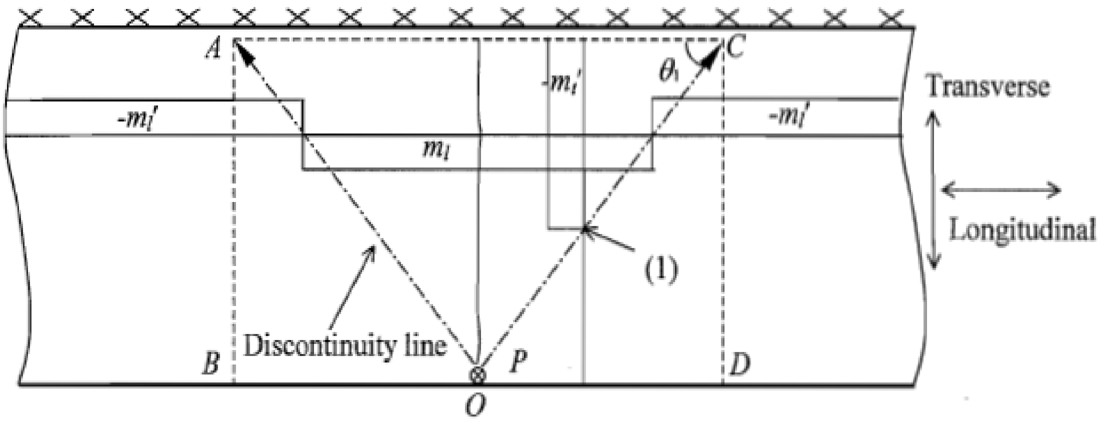

Fig. 6 Torqueless grillage solution

This value is independent of the position of the load, as is clear on dimensional arguments. It does depend on the assumption that the bridge is long, so that the hogging moment m_l' can be developed - it is possible to investigate by the strip method the minimum length of bridge (as a multiple of the distance of the load from the cantilever root). It is also possible to develop other lower bounds by considering additional discontinuity lines coming from the load towards the edge of the cantilever, and thence turning back towards the root; and to take into account the bending strength of an edge beam [20].

Having explicit lower bound solutions for such a point load seems to be an interesting development. The question arises whether these solutions are at all close to the upper bounds on flexural failure derived by yield line theory, a matter also investigated by Lu. It seems not to be easy to extend this approach to deal with load spread over a small patch, though one presumes that equation (1) would also give a lower bound on the total load in that case.

However, a more fundamental question remains – whether it is reasonable in practice to push this bending theory so far, and rely on infinite shear strength, especially when the load can be placed anywhere and so shear reinforcement such as stud rails cannot be provided. Perhaps this is not a lower bound at all, whatever its theoretical interest.

6 IMPROVING DUCTILITY IN TENSION

It is well known that the addition of short (order 20 mm) steel fibres to the concrete mix, in volume proportions up to a practical limit of about 1.5%, will greatly increase the ductility of the material, though with little if any increase in strength. The ductility seems to arise not so much from plastic flow in the high-strength steel fibres, as from gradual pull-out of the fibres from the matrix. With the larger proportions of fibres, and the corresponding ductility in tension as well as compression, one might expect plasticity theory to be more widely applicable than for plain concrete.

Suspended slabs in buildings are being constructed with little or no ordinary reinforcement, but with steel fibres instead. For bending failure and yield line theory one would expect to be able to predict the collapse load (in the absence of membrane action) by estimating the bending strength m_u developed on yield lines. In plastic theory this estimate would be made using rectangular stress blocks, one in compression presumably at the usual 0.6 f_{cd} and one in tension at the split-cylinder tensile strength of the matrix, perhaps with an effectiveness factor. This bears some resemblance to recent recommendations [21].

What of other failure modes where plastic theory for plain concrete has been less successful? There is a plastic theory, due to Braestrup [22], for punching shear failure under concentrated loads, with provision for non-zero tensile strength. In recent years students in Cambridge have investigated whether this theory is applicable to small-size circular slabs in punching shear, with different volume proportions of steel fibres. In 2002 Kiilu [23] tested 66 mm thick slabs of 1000mm diameter in high-strength concrete (about 105 MPa on cubes), simply-supported at 940 mm diameter, with load applied to a central stub column of 80 mm diameter (Fig.7). In 2007 Pears [24] tested slabs of similar dimensions but with cube strengths of order 65 MPa. In both cases, heavy bar reinforcement was provided in both directions, to eliminate any possibility of failure by bending in a conical mechanism with a fan of yield lines, and thus to enforce failure by punching shear near the central stub column. The boundary conditions prevented membrane forces at the edges, though they may have developed near the centre, equilibrated by forces in the outer unfailing part of the slab.

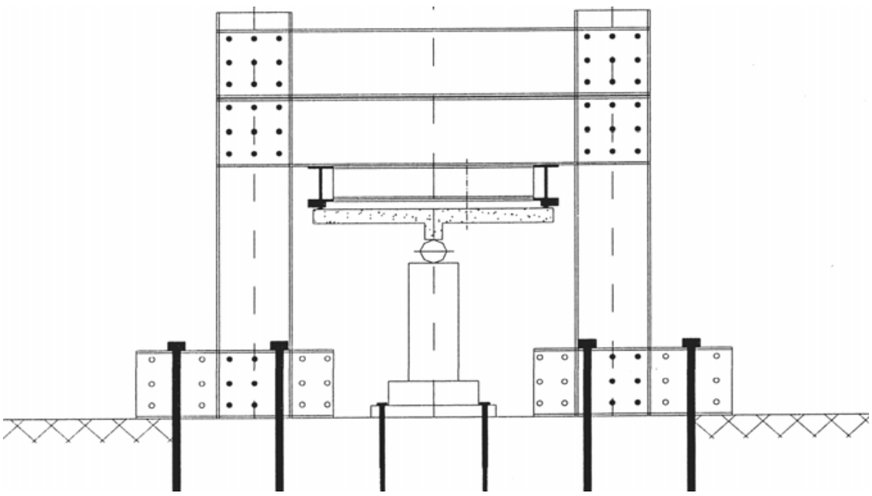

Fig. 7 Rig for punching shear tests [23]

It is possible to modify the plastic theory (Kuang [25] and Kiilu [23]), by smoothing the Mohr-Coulomb failure surface with tension cut-off, to produce a simpler theory, to predict such things as the shape of the axisymmetric failure surface, and the value of the collapse load. However, the question at issue is not so much whether plastic theory of this kind can be made to give reasonable predictions – with suitable effectiveness factors it often will – as whether the failure is in fact ductile as the theory requires, so that it might be applicable at larger size. Plots of load against central deflection for Kiilu's tests are given in Fig. 8, where it is apparent that the slabs with more fibres exhibited some ductility, prior to a sudden failure.

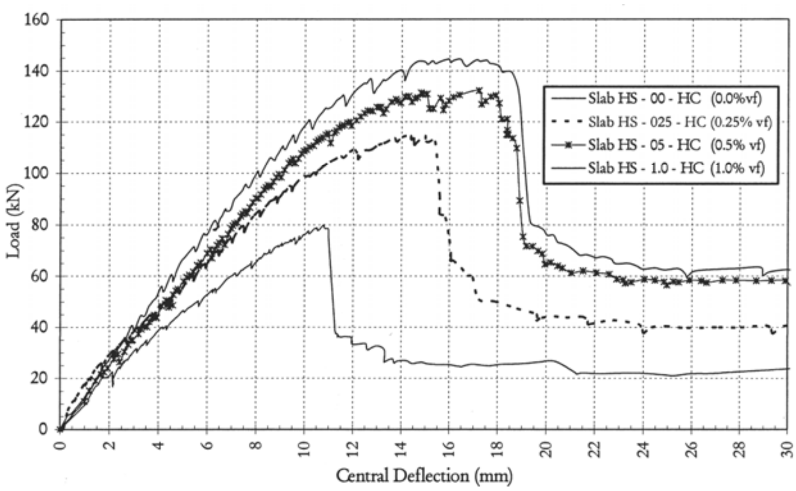

Fig. 8 Applied load against central deflection [23]

For his first slab, with 0.5% fibres, Pears measured the displacement profile along a radius of the slab at different loads (Fig.9) – and found little sign of shear displacement until after the failure, which occurred rather suddenly (although the test was deflection-controlled, in a stiff rig with a screw jack applying load). In subsequent tests, with 0.2 and 1.0 % of fibres, Pears measured the displacement of the column stub relative to the adjacent part of the slab, to see whether the expected shear deformation was occurring gradually. Only in the slab with 1% of fibres (Fig. 10) was there much sign of shear displacement developing gradually – i.e. much sign of ductility in shear. Both slabs failed rather suddenly, with a rapid drop in applied load, and apparently with appreciable unloading of elastic strain energy into the failure region. This needs to be further investigated, if plastic theory is to be applied to such punching shear failures, even in slabs with high volume fractions of fibres and so appreciable ductility in tension, especially in slabs of greater thickness.

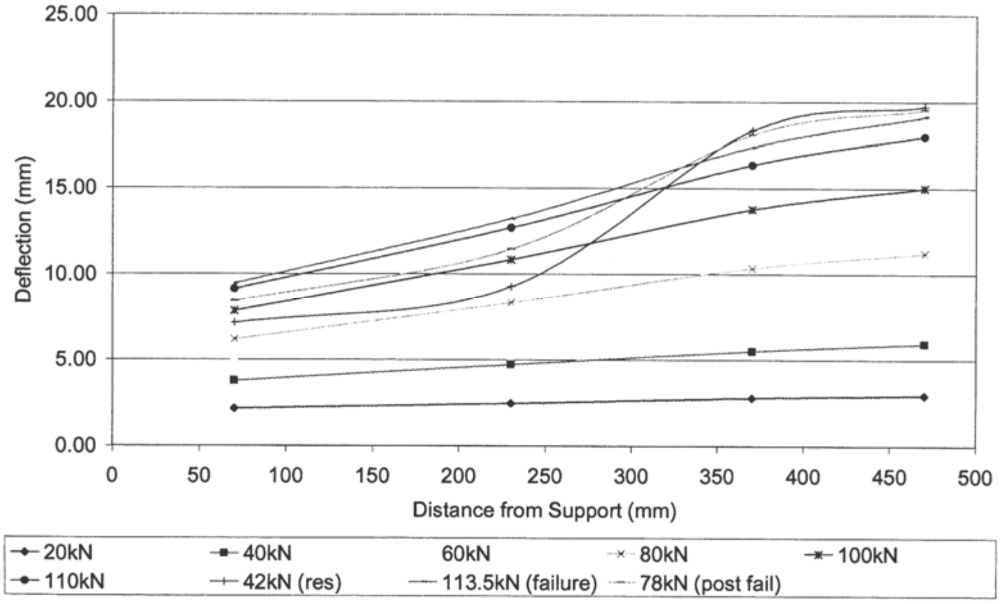

Fig. 9 Deflection patterns at different load stages – 0.5% fibres [24]

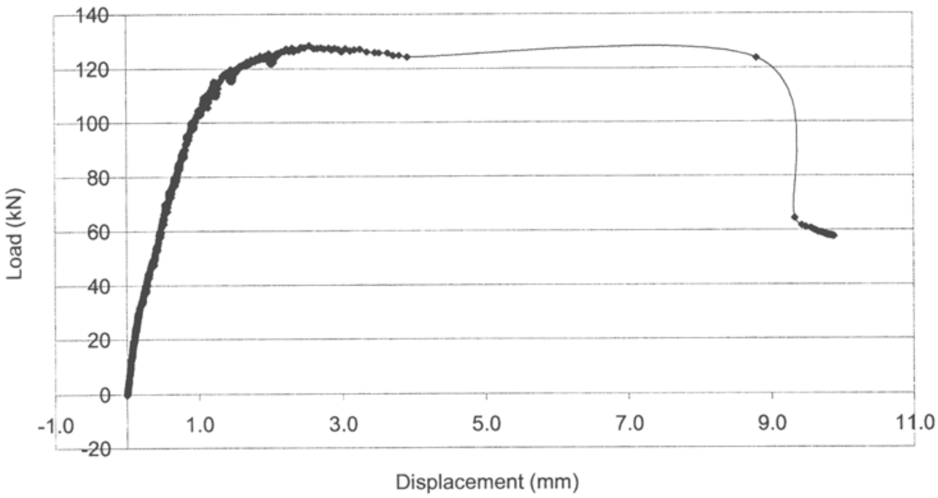

Fig. 10 Deflection of column stub relative to adjacent slab – 1.0 % fibres [24]

7 ALTERNATIVE THEORIES ?

Another question that arises, if plasticity theory is found wanting at least for some modes of failure, is what alternative theories could be used to predict collapse loads, especially of large concrete structures. An obvious possibility is some form of non-linear computerised analysis, perhaps using finite elements (NLFEA). One problem here is to establish the material properties etc needed for structures of larger size than the program has been verified upon. Another problem is that abandoning plasticity theory also means abandoning one comfort it brings, namely that residual stresses have no effect on the collapse load. So with NLFEA one presumably ought to start the analysis not only with zero stress throughout – as is common – but also with a whole range of conceivable residual stress (and crack?) patterns, due perhaps to temperature change or support settlement, before increasing the applied load towards failure.

8 CONCLUSIONS

Plasticity theory can justifiably play a significant part in analysis and design of concrete structures, and form a useful basis for parts of codes of practice, especially when suitably modified and applied primarily to structures of ordinary type and size of which there is much experience. However, its application to non-ductile failure modes is distinctly problematic, and there is need for great care and wariness in attempting to extrapolate its use to unusual structures and particularly to structures of large size without adequate ductility.

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